PROJECTION METHODS AND SELF-REGULARIZATION IN ILL-POSED PROBLEMS

G. M. Vainikko and U. A. Khyamarik

Izvestiya VUZ. Matematika,
Vol. 29, No. 10, pp. 3-17, 1985

UDC 519.642

The stable solution of ill-posed problems, as is well known, is attained using A. N. Tikhonov's regularization method, iteration or some other special regularization methods [1-7]. When preparing the problem for solution on a computer its discretization is unavoidable. It has been noted that a successful discretization sometimes has additional regularizing properties. It was discovered, moreover, that when solving certain types of problem we can avoid discretization, omitting the regularization step (in this case we refer to the self-regularization of the problem when it is discretized, see [8]). Volterra's integral equations of the first kind, for which the methods of quadrature sums and close algorithms are examined, serve as an example (see, for example, [9, 10] and the reviews in [11,12,6]).

This review analyzes self-regularization when discretizing a problem using projection methods. The regularizing properties of some specific methods (the least squares, least error and Galerkin methods) were examined in [13-28]. We shall set ourselves a broader goal – to show the general conditions under which projection methods lead to the self-regularization of an ill-posed problem. Note that self-regularization in other algorithms for the discretization of ill-posed problems is examined in [29-37,25], and the discretization of regularizing algorithms, for example, is examined in [2,5,6,38-41,28].

1. SOLUTION OF OPERATOR EQUATIONS

1.1 Formulation of the problem. Suppose H and F are Hilbert spaces, $A\in\mathcal{L}(H, F)$, i.e., A is a linear continuous operator from H to F. We will assume that the area of the values R(A) of operator A is open in F, and thus the problem of calculating $u\in H$ from equation

$$Au=f \tag{1.1}$$

is incorrectly formulated. We assume that the right-hand side $f\in R(A)$ of Eq. (1.1) is inaccurately known, and instead of it there is some element $f_\delta \in F$, $\|f_\delta - f\| \leq \delta$ in our arrangement. We can also consider the case when operator A is also approximately known, but to simplify the calculations we will only refer to this case in section 1.10.

To solve Eq. (1.1) we will use the projection method

$$u_n \in H_n, \ \langle Au_n - f_\delta, \ v_n \rangle = 0 \ (\forall v_n \in F_n). \tag{1.2}$$

Here $H_n \subset H$ and $F_n \subset F$ $(n = 1, 2, \ldots)$ are finite-dimensional subspaces,

$$\dim H_n = \dim F_n;$$

we shall denote the corresponding orthoprojectors by P_n and Q_n $(P_n H = H_n, \ Q_n F = F_n)$. Conditions (1.2) are equivalent to equation

$$A_n u_n = Q_n f_\delta, \ \ A_n = Q_n A P_n \in \mathcal{L}(H_n, \ F_n). \tag{1.3}$$

We are interested in the problem of the conditions under which the solutions u_n of Eqs. (1.3) as $\delta \to 0$, $n = n(\delta) \to \infty$ approach the solution of Eq. (1.1).

1.2. Subsidiary results. We shall denote the zero subspace (kernel) of th linear operator B by N(B).

Lemma 1.1. Suppose $N(A) \cap H_n = 0$ and $\tau_n \equiv \inf_{v_n \in H_n} \|Q_n A v_n\| / \|A v_n\| > 0$. Then we shall invert $A_n \in \mathcal{L}(H_n, F_n)$ and $\varkappa_n \leq \|A_n^{-1}\| \leq \varkappa_n / \tau_n$, where

$$\varkappa_n = \sup_{v_n \in H_n} \|v_n\| / \|A v_n\|. \tag{1.4}$$

Proof. For any $w_n \in H_n$ we have

$$\|A_n w_n\| \geq \tau_n \|A w_n\| \geq \tau_n \varkappa_n^{-1} \|w_n\|;$$

for the element $v_n \in H_n$, in which the maximum is attained in (1.4), we have

$$\|A_n v_n\| \leq \|A v_n\| = \varkappa_n^{-1} \|v_n\|.$$

Hence follow the statements of the lemma.

Lemma 1.2. Suppose $N(A^*) \cap F_n = 0$ and $\tau_n^* \equiv \inf_{z_n \in F_n} \|P_n A^* z_n\| / \|A^* z_n\| > 0$. Then we will invert $A_n \in \mathcal{L}(H_n, F_n)$ and

$$\varkappa_n^* \leq \|A_n^{-1}\| \leq \varkappa_n^* / \tau_n^*, \ \|A_n^{-1} Q_n A\| = 1/\tau_n^*, \ \|A_n^{-1} Q_n A (I - P_n)\| = \sigma_n^*,$$

where $\sigma_n^* = \sup_{z_n \in F_n} \|(I - P_n) A^* z_n\| / \|P_n A^* z_n\|$,

$$\varkappa_n^* = \sup_{z_n \in F_n} \|z_n\| / \|A^* z_n\|, \tag{1.5}$$

$$1/\tau_n^* = [1 + (\sigma_n^*)^2]^{1/2}. \tag{1.6}$$

Proof. The first statement is proved in a similar way to Lemma 1.1 – the following operator is adjoint to $A_n = Q_n A P_n \in \mathcal{L}(H_n, F_n)$:

$$A_n^* = P_n A^* Q_n \in \mathcal{L}(F_n, H_n).$$

Further, $\|A_n^{-1} Q_n A\| = \lambda_n^{1/2}$, where λ_n is the largest eigenvalue of the operator $A_n^{-1} Q_n A (A_n^{-1} Q_n A)^* \in \mathcal{L}(H_n, H_n)$; suppose v_n is the corresponding eigenelement: $A_n^{-1} Q_n A A^* Q_n (A_n^*)^{-1} v_n = \lambda_n v_n$, $v_n \in H_n$. Then

$$Q_n A A^* Q_n x_n = \lambda_n A_n A_n^* x_n, \quad x_n = (A_n^*)^{-1} v_n \in F_n,$$

and λ_n also remains the largest eigenvalue for the latter problem. The variational characteristics of the largest eigenvalue give

$$\lambda_n = \sup_{z_n \in F_n} \frac{\langle Q_n A A^* Q_n z_n, z_n \rangle}{\langle A_n A_n^* z_n, z_n \rangle} = \sup_{z_n \in F_n} \frac{\|A^* z_n\|^2}{\|P_n A^* z_n\|^2} = 1/(\tau_n^*)^2,$$

and the equation $\|A_n^{-1} Q_n A\| = 1/\tau_n^*$ is established. The equation $\|A_n^{-1} Q_n A (I - P_n)\| = \sigma_n^*$ is established in a similar way. We will prove (1.6):

$$\left(\frac{1}{\tau_n^*}\right)^2 = \sup_{z_n \in F_n} \frac{\|A^* z_n\|^2}{\|P_n A^* z_n\|^2} = \sup_{z_n \in F_n} \frac{\|P_n A^* z_n\|^2 + \|(I - P_n) A^* z_n\|^2}{\|P_n A^* z_n\|^2} =$$
$$= \sup_{z_n \in F_n} \left(1 + \frac{\|(I - P_n) A^* z_n\|^2}{\|P_n A^* z_n\|^2}\right) = 1 + (\sigma_n^*)^2.$$

Remark. The conditions of Lemmas 1.1 and 1.2 are equivalent. At the same time

$$\tau_n^* \geq \tau_n/(\tau_n^2 + \varkappa_n^2 \varepsilon_n^2)^{1/2}, \quad \tau_n \geq \tau_n^*/(\tau_n^{*2} + \varkappa_n^{*2} \varepsilon_n^{*2})^{1/2},$$
$$\varepsilon_n = \|(I - P_n) A^* Q_n\|, \quad \varepsilon_n^* = \|(I - Q_n) A P_n\|. \tag{1.7}$$

In fact, the conditions of each of Lemmas 1.1 and 1.2 are equivalent to the invertibility of A_n. From the relations

$$\sigma_n^* = \|A_n^{-1} Q_n A (I - P_n)\| \leq \|A_n^{-1}\| \|Q_n A (I - P_n)\| \leq \varkappa_n \varepsilon_n / \tau_n$$

using (1.6) we obtain the first inequalities (1.7); the second is obtained from considerations of duality.

1.3. The a priori choice of dimensions.

Theorem 1.1. Suppose $A \in \mathcal{L}(H, F)$, $f \in R(A)$, $\|f_\delta - f\| \leq \delta$,

$$\|u - P_n u\| \to 0 \quad \text{as} \quad n \to \infty \ (\forall u \in H), \tag{1.8}$$

$$N(A^*) \cap F_n = 0 \ (n \geq n_0), \tag{1.9}$$

$$\|P_n A^* z_n\| \geq \tau^* \|A^* z_n\| \ (\forall z_n \in F_n, \ n \geq n_0), \ \tau^* = \text{const} > 0. \tag{1.10}$$

Then Eq. (1.1) has the unique solution $u_* \in H$, and Eq. (1.3) when $n \geq n_0$ is the unique solution $u_n \in H_n$. Choosing $n = n(\delta)$, such that

$$n(\delta) \to \infty, \ \delta \varkappa_{n(\delta)}^* \to 0 \quad \text{as} \quad \delta \to 0 \tag{1.11}$$

(see (1.5)), we obtain the convergence $\|u_{n(\delta)} - u_*\| \to 0$ as $\delta \to 0$.

Proof. The unique solvability of Eq. (1.3) follows from (1.9), (1.10) and

Lemma 1.2. The solvability of Eq. (1.1) follows from the condition $f \in R(A)$; to prove the uniqueness of the solution we need to show that N(A) = 0. Discussing the opposite, we will assume that for some $u_0 \neq 0$ we have $Au_0 = 0$. When $u_0 \neq 0$ the equation $A_n^* z_n = P_n u_0$ has the unique solution $z_n \in F_n$; at the same time $P_n A^* z_n = P_n u_0 \to u_0$ as $n \to \infty$. By virtue of (1.10)

$$\|A^* z_n\| \leqslant (1/\tau^*) \| P_n A^* z_n \| \leqslant \text{const},$$

therefore the sequence $A^* z_n$ contains a weakly converging sequence. Suppose $A^* z_n \to v$. From (1.8) it follows that $P_n A^* z_n \to v$, and this means, $v = u_0$. Thus, $A^* z_n \to u_0$, $\langle A^* z_n, u_0 \rangle \to \|u_0\|^2$. But since $\langle A^* z_n, u_0 \rangle = \langle z_n, Au_0 \rangle = 0$, and $u_0 \neq 0$, we arrive at a contradiction.

We shall proceed to prove the convergence of the approximations u_n. From the equation $A_n(u_n - P_n u_*) = Q_n(f_\delta - f) + Q_n A(u_* - P_n u_*)$ on the grounds of Lemma 1.2 we will obtain

$$\|u_n - u_*\| \leqslant \|A_n^{-1}\|\delta + [1 + \|A_n^{-1} Q_n A(I - P_n)\|]\|u_* - P_n u_*\| \leqslant$$
$$\leqslant \frac{\varkappa_n^*}{\tau_n^*}\delta + (1 + \delta_n^*)\|u_* - P_n u_*\| \leqslant \frac{\varkappa_n^*}{\tau^*}\delta + \left[1 + \frac{(1 - \tau^{*2})^{1/2}}{\tau^*}\right]\|u_* - P_n u_*\| \, (n \geqslant n_0) \qquad (1.12)$$

bearing in mind (1.6) and the inequality $\tau_n^* \geqslant \tau^*$ that follows from (1.10). Hence on the ground of (1.8) and (1.11) we will obtain that $u_{n(\delta)} \to u_*$ as $\delta \to 0$. Theorem 1.1 is proved.

Remark 1. Under the conditions of Theorem 1.1 we have a self-regularization of problem (1.1). In other words, the projection method (1.2) with the choice of dimensions in accordance with (1.11) determines the regularizator of problem (1.1).

Remark 2. As is obvious from the proof, we can replace the condition $\|f_\delta - f\| \leqslant \delta$ in Theorem 1.1 by the condition $\|Q_n(f_\delta - f)\| \leqslant \delta$. This remark concerns the remaining theorems presented below.

Remark 3. Conditions (1.8)-(1.10) are necessary for the existence of the connection $n = n(\delta)$, such that $u_{n(\delta)} \to u_* = A^{-1}f$ as $\delta \to 0$ for any $f \in R(A)$, of the approximately known ($\|f_\delta - f\| \leqslant \delta$). In the case of an accurately specified right-hand side of ($f_\delta = f$) conditions (1.8)-(1.10) are necessary and sufficient for the convergence $u_n \to u_*$ as $n \to \infty$ for any $f \in R(A)$.

Remark 4. From (1.8)-(1.10) and the openness R(A) it follows that $\varkappa_n \to \infty$, $\varkappa_n^* \to \infty$ as $n \to \infty$.

1.4. Choice of dimensions with respect to the discrepancy.

Lemma 1.3. Suppose the conditions of Lemma 1.1 hold. Then for the discrepancy of the solution of Eq. (1.3) the following inequality holds:

$$\|Au_n - f_\delta\| \leqslant \tau_n^{-1}\,\mathrm{dist}(f_\delta,\ AH_n). \tag{1.13}$$

Proof. We shall denote the orthoprojector in F, projecting to AH_n, by Q'_n. We have (see [42]): $\|(I-Q_n)Q'_n\| \leqslant \|Q_n - Q'_n\| = (1-\tau_n^2)^{1/2}$. Since to solve Eq. (1.3) $Q'_n(Au_n - f_\delta) = 0$, and $Q'_nAu_n = Au_n$, then

$$[\mathrm{dist}(f_\delta,\ AH_n)]^2 = \|f_\delta - Q'_nf_\delta\|^2 = \|(f_\delta - Au_n) - Q'_n(f_\delta - Au_n)\|^2 =$$
$$= \|f_\delta - Au_n\|^2 - \|Q'_n(f_\delta - Au_n)\|^2 = \|f_\delta - Au_n\|^2 - \|Q'_n(I-Q_n)(f_\delta - Au_n)\|^2 \geqslant$$
$$\geqslant \|f_\delta - Au_n\|^2 - (1-\tau_n^2)\|f_\delta - Au_n\|^2 = \tau_n^2\|f_\delta - Au_n\|^2,$$

which it is also required to establish.

Theorem 1.2. Suppose $A \in \mathcal{L}(H,\ F)$, $f \in R(A)$, $\|f_\delta - f\| \leqslant \delta$,

$$\|u - P_nu\| \to 0 \quad \text{as} \quad n \to \infty \ (\forall u \in H), \tag{1.14}$$

$$N(A) \cap H_n = 0 \ (n \geqslant n_0), \tag{1.15}$$

$$\|Q_nAv_n\| \geqslant \tau\|Av_n\| \ (\forall v_n \in H_n,\ n \geqslant n_0),\ \tau = \mathrm{const} > 0, \tag{1.16}$$

$$\|P_nA^*z_n\| \geqslant \tau^*\|A^*z_n\| \ (\forall z_n \in F_n,\ n \geqslant n_0),\ \tau^* = \mathrm{const} > 0, \tag{1.17}$$

$$\varkappa_{n+1}\|(I - Q'_n)A\| \leqslant \gamma' = \mathrm{const} \ (n \geqslant n_0), \tag{1.18}$$

where κ_n is a quantity determined in (1.4), and Q'_n is the orthoprojector in F to AH_n. Then Eq. (1.1) has the unique solution $u_* \in H$, and Eq. (1.3) when $n \geqslant n_0$ has the unique solution $u_n \in H_n$. If for $n = n(\delta)$ we choose the first of the numbers n = 1, 2, ..., for which Eq. (1.3) is solvable and

$$\|Au_n - f_\delta\| \leqslant b\delta,\ b = \mathrm{const},\ b > \tau^{-1}, \tag{1.19}$$

then we have the convergence $\|u_{n(\delta)} - u_*\| \to 0$ as $\delta \to 0$.

Proof. The unique solvability of Eq. (1.3) when $n \geqslant n_0$ follows from conditions (1.15) and (1.16). The solvability of Eq. (1.1) follows from the condition $f \in R(A)$, and the uniqueness of the solution is established in the same way as in Theorem 1.1. From the proof of Theorem 1.1 we have estimate

$$\|u_n - u_*\| \leqslant \frac{\varkappa_n}{\tau}\delta + \left[1 + \frac{(1-\tau^{*2})^{1/2}}{\tau^*}\right]\|u_* - P_nu_*\| \ (n \geqslant n_0) \tag{1.20}$$

(comp. (1.12); Lemma 1.1 was used when estimating the norm $\|A_n^{-1}\|$).

Using Lemma 1.3 $\overline{\lim_{n\to\infty}}\|Au_n - f_\delta\| \leqslant \tau^{-1}\delta$, therefore the choice $n = n(\delta)$, shown in the formulation of the theorem, can be implemented. Consider first the basic case when $n(\delta) \to \infty$ as $\delta \to 0$. For $m = m(\delta) = n(\delta) - 1$ we have $\|Au_m - f_\delta\| > b\delta$, and using Lemma 1.3 $b\delta < \|Au_m - f_\delta\| \leqslant \tau^{-1}\,\mathrm{dist}(f_\delta,\ AH_m) \leqslant \tau^{-1}[\delta + \mathrm{dist}(f,\ AH_m)]$. Since $b > \tau^{-1}$, then

$$\delta \leqslant \frac{1}{b\tau - 1}\mathrm{dist}(f,\ AH_m) = \frac{1}{b\tau - 1}\|Au_* - Q'_mAu_*\| = \frac{1}{b\tau - 1}\|(I - Q'_m)A(I - P_m)u_*\|.$$

Using condition (1.18)

$$\frac{\varkappa_n}{\tau}\delta < \frac{\varkappa_n}{(b\tau-1)\tau}\|(I-Q'_{n-1})A\|\|(I-P_{n-1})u_*\| < \frac{\gamma'}{(b\tau-1)\tau}\|(I-P_{n-1})u_*\|,$$

and estimate (1.20) when $n=n(\delta)$, determined using the discrepancy, takes the form

$$\|u_n-u_*\| < \frac{\gamma'}{(b\tau-1)\tau}\|u_*-P_{n-1}u_*\| + \left[1+\frac{(1-\tau^{*2})^{1/2}}{\tau^*}\right]\|u_*-P_nu_*\|. \qquad (1.21)$$

Hence it follows that $u_{n(\delta)}\to u_*$ as $\delta\to 0$.

If for some $\delta_k\to 0$ the rule for choosing n shown in the theorem will issue $n(\delta_k)\leq l=\text{const}\,(k=1,2,\ldots)$, then estimate (1.21) does not enable us to judge the convergence $u_{n(\delta)}\to u_*$. But the sequence $u_{n(\delta_k)}$, $k=1,2,\ldots$, in that case will lie in a finite-dimensional subspace - the linear envelope of the subspaces H_n, $n=1,\ldots,l$; from (1.20) it follows that it is bounded, and therefore (relatively) compact in H. Since $\|Au_{n(\delta)}-f_\delta\|\leq b\delta$, then $Au_{n(\delta_k)}\to f=Au_*$ as $k\to\infty$. Hence it follows that $u_{n(\delta_k)}\to u_*$ as $k\to\infty$. Theorem 1.2 is proved.

Remark 1. If the conditions of Theorem 1.2 hold and $H_n\subset H_{n+1}$ $(n=1,2,\ldots)$, then we have the inequality $n(\delta)\leq l=\text{const}\,(\delta\to 0)$ when, and only when, u_* belongs to one of the subspaces H_n.

Remark 2. Suppose E is some Banach space, into which H is continuously inserted; we shall formulate $M_\rho=\{u\in H:\|u\|_H\leq\rho\}\subset E$. It appears that under the conditions of Theorem 1.2 the projection method (1.2) with the choice $n=n(\delta)$ using the discrepancy (shown in the theorem by the form) determines the order-optimal method of solving problem (1.1) in M_ρ, when the error is measured using the norm of the space E:

$$\sup_{\substack{u\in M_\rho,\, f_\delta\in F\\ \|Au-f_\delta\|\leq\delta}}\|u_{n(\delta)}-u\|_E\leq c\inf_P\sup_{\substack{u\in M_\rho,\, f_\delta\in F\\ \|Au-f_\delta\|\leq\delta}}\|Pf_\delta-u\|_E,$$

$$c=1+\max\left\{b,\ \frac{\gamma'}{(b\tau-1)\tau}+\frac{1}{\tau^*}\right\}$$

(the infimum is taken using the whole mapping $P:F\to E$). For more detail about the concent of optimality see [2].

Remark 3. Suppose $\theta\in(0,1)$. Replacing condition (1.18) by

$$\varkappa_n\max_{\theta n\leq m\leq n}\|(I-Q'_m)A\|\leq\gamma'=\text{const}\,(n=1,2,\ldots),$$

Theorem 1.2 will also remain valid in the case when any of n = 1, 2, ..., for which (1.19) holds and for which $m\in[\theta n, n]$ exists with the property $\|Au_m-f_\delta\|>b\delta$ is taken for $n(\delta)$.

1.5. Additions to the theorems of convergence. We shall indicate some cases when, from the satisfaction of one of the conditions (1.16) and (1.17), follows the satisfaction of another.

Lemma 1.4. If (1.16) holds and $\varkappa_n\|(I-P_n)A^*Q_n\| \leq \gamma = \text{const}\ (n \geq n_0)$, then (1.17) holds with $\tau^* = \tau(\tau^2+\gamma^2)^{-1/2}$. Similarly, if (1.17) holds and $\varkappa_n^*\|(I-Q_n)AP_n\| \leq \gamma^* = \text{const}\ (n \geq n_0)$, then (1.16) holds with $\tau = \tau^*(\tau^{*2}+\gamma^{*2})^{-1/2}$.

The proof directly follows from (1.7).

Lemma 1.5. If (1.16) holds and for some $\alpha \in (0, 1]$

$$\varkappa_n^\alpha\|(I-P_n)(A^*A)^{\alpha/2}\| \leq \gamma = \text{const}\ (n \geq n_0), \tag{1.22}$$

then (1.17) holds with $\tau^* = \tau(\tau^{2\alpha}+\gamma^2)^{-1/(2\alpha)}$. Similarly, if (1.17) holds and

$$(\varkappa_n^*)^\alpha\|(I-Q_n)(AA^*)^{\alpha/2}\| \leq \gamma^* = \text{const}\ (n \geq n_0), \tag{1.23}$$

then (1.16) holds with $\tau = \tau^*[(\tau^*)^{2\alpha}+(\gamma^*)^2]^{-1/(2\alpha)}$.

Proof. We will use the polar expansion [43] of operator A in the form $A=(AA^*)^{1/2}U$, where $U \in \mathcal{L}(H, F)$ is a partially isometric operator, $\|Uv\| \leq \|v\|\ (\forall v \in H)$. Formulating $B=(AA^*)^{1/2} \in \mathcal{L}(F, F)$, we have

$$\frac{1}{\tau_n^2} = \sup_{v_n \in H_n} \frac{\|Av_n\|^2}{\|Q_nAv_n\|^2} = \sup_{z_n \in UH_n} \frac{\|Bz_n\|^2}{\|Q_nBz_n\|^2} = 1 + \sup_{z_n \in UH_n} \frac{\|(I-Q_n)Bz_n\|^2}{\|Q_nBz_n\|^2} \leq$$

$$\leq 1 + \|(I-Q_n)B^\alpha\|^2 \sup_{z_n \in UH_n} \frac{\|B^{1-\alpha}z_n\|^2}{\|Q_nBz_n\|^2}.$$

Using the inequality of the aspects [42]: $\|B^{1-\alpha}z_n\| \leq \|Bz_n\|^{1-\alpha}\|z_n\|^\alpha$ the latter inequality acquires the form

$$\frac{1}{\tau_n^2} \leq 1 + \|(I-Q_n)B^\alpha\|^2 \frac{1}{\tau_n^{2(1-\alpha)}}\left(\sup_{v_n \in H_n} \frac{\|v_n\|}{\|Q_nAv_n\|}\right)^{2\alpha} = 1 + \|(I-Q_n)B^\alpha\|^2 \frac{1}{\tau_n^{2(1-\alpha)}}\|A_n^{-1}\|^{2\alpha}.$$

If (1.17) holds, then using Lemma 1.2 $\|A_n^{-1}\| \leq \varkappa_n^*/\tau_n^* \leq \varkappa_n^*/\tau^*\ (n \geq n_0)$, and the inequality - bearing in mind condition (1.23) - is transformed to the form

$$\frac{1}{\tau_n^2} \leq 1 + \frac{\gamma^{*2}}{(\tau^*)^{2\alpha}\tau_n^{2(1-\alpha)}}\ (n \geq n_0).$$

Hence the boundedness of the sequence $1/\tau_n$ is obvious; a more thorough analysis shows that $\tau_n \geq \tau^*[(\tau^*)^{2\alpha}+\gamma^{*2}]^{-1/(2\alpha)}\ (n \geq n_0)$. This proves the second statement of the lemma. The first statement is proved in a similar way.

Lemma 1.6. If $\varkappa_n^*\|(I-P_n)A^*Q_n\| \leq q < 1\ (n \geq n_0)$., condition (1.17) holds with $\tau^* = (1-q^2)^{1/2}$. Similarly, if $\varkappa_n\|(I-Q_n)AP_n\| \leq q < 1$, then (1.16) holds with $\tau = (1-q^2)^{1/2}$.

Proof. The second statement is proved thus:

$$\|Q_nAv_n\|^2 = \|Av_n\|^2 - \|(I-Q_n)Av_n\|^2 \geq \|Av_n\|^2 - \left(\frac{q}{\varkappa_n}\|v_n\|\right)^2 \geq (1-q^2)\|Av_n\|^2.$$

The first statement is proved in a similar way.

Lemma 1.7. Suppose $H=F,\ H_n=F_n,\ A=A^*>0$. If for some $\alpha \in (0, 1]$

$$\varkappa_n^\alpha\|(I-P_n)A^\alpha\| \leq \gamma = \text{const}\ (n=1, 2, \ldots),$$

then condition (1.16) holds (which in this case agrees with (1.17)), and $n_0 = 1$, $\tau = (1+\gamma^2)^{-1/(2\alpha)}$ when $\alpha \in (0, 1/2]$, $\tau = (1+\gamma)^{-1/\alpha}$ when $\alpha \in (1/2, 1]$.

Proof. In the same way as when proving Lemma 1.5 we establish the inequality

$$\frac{1}{\tau_n^2} \leqslant 1 + \|(I-P_n)A^\alpha\|^2 \sup_{v_n \in H_n} \frac{\|A^{1-\alpha}v_n\|^2}{\|P_n A v_n\|^2}.$$

Suppose $\alpha \in (0, 1/2]$. Using the inequality of the aspects $\|A^{1-\alpha}v_n\| \leqslant \|Av_n\|^{1-2\alpha}\|A^{1/2}v_n\|^{2\alpha}$. Bearing in mind, also, that $\|A^{1/2}v_n\|^2 = (P_n A v_n, v_n) \leqslant \|P_n A v_n\|\|v_n\|$, we obtain

$$\frac{1}{\tau_n^2} \leqslant 1 + \|(I-P_n)A^\alpha\|^2 \sup_{v_n \in H_n} \frac{\|Av_n\|^{2(1-2\alpha)}}{\|P_n A v_n\|^{2(1-2\alpha)}} \sup_{v_n \in H_n} \frac{\|v_n\|^{2\alpha}}{\|P_n A v_n\|^{2\alpha}} \leqslant$$
$$\leqslant 1 + \|(I-P_n)A^\alpha\|^2 \frac{1}{\tau_n^{2(1-2\alpha)}} \|A_n^{-1}\|^{2\alpha}.$$

Since $\|A_n^{-1}\| \leqslant \varkappa_n/\tau_n$, then bearing in mind the condition of the lemma we finally obtain the inequality $1/\tau_n^2 \leqslant 1 + \gamma^2/\tau_n^{2(1-\alpha)}$ $(n \geqslant 1)$, whence the statement of the lemma for $\alpha \in (0, 1/2]$ also follows. Since $\|(I-P_n)A^{\alpha/2}\|^2 \leqslant \|(I-P_n)A^\alpha\|$, the case $\alpha \in (1/2, 1]$ reduces to the case $\alpha \in (0, 1/2]$.

Lemma 1.8. For any natural m the following inequality holds:

$$\|(I-Q_n')A\| \leqslant \|(I-P_n)(A^*A)^{1/(2m)}\|^m. \qquad (1.24)$$

Thus, for condition (1.18) to be satisfied, it is sufficient that

$$\varkappa_{n+1}^{1/m}\|(I-P_n)(A^*A)^{1/(2m)}\| \leqslant \text{const} \ (n \geqslant n_0). \qquad (1.25)$$

Proof. We will use the polar expansion $A = U(A^*A)^{1/2}$. We will denote the orthoprojector in $(A^*A)^{1/2}H_n$ by P'_n. Since $UP_n'H \subset AH_n$, then $(I-Q_n')UP_n' = 0$ and

$$(I-Q_n')A = (I-Q_n')U(I-P_n')(A^*A)^{1/2}, \ \|(I-Q_n')A\| \leqslant \|(I-P_n')(A^*A)^{1/2}\|.$$

We shall denote the orthoprojector in $(A^*A)^{j/(2m)}H_n$, $j = 0, 1, \ldots, m$ by $P_n^{(j)}$; in particular, $P_n^{(0)} = P_n$, $P_n^{(m)} = P_n'$. Obviously, $(I-P_n^{(j)})(A^*A)^{1/(2m)}P_n^{(j-1)} = 0$, $j = 1, \ldots, m$. Hence follows

$$(I-P_n')(A^*A)^{1/2} = (I-P_n^{(m)})(A^*A)^{1/(2m)}(I-P_n^{(m-1)})(A^*A)^{1/(2m)} \ldots (I-P_n^{(1)})(A^*A)^{1/(2m)},$$
$$\|(I-P_n')(A^*A)^{1/2}\| \leqslant \prod_{j=1}^m \|(I-P_n^{(j)})(A^*A)^{1/(2m)}\|.$$

From the equation $(I-P_n^{(j)})(A^*A)^{1/(2m)} = (I-P_n^{(j)})(A^*A)^{1/(2m)}(I-P_n^{(j-1)})$ we will obtain

$$\|(I-P_n^{(j)})(A^*A)^{1/(2m)}\| \leqslant \|(I-P_n^{(j-1)})(A^*A)^{1/(2m)}\| \leqslant \ldots \leqslant \|(I-P_n)(A^*A)^{1/(2m)}\|,$$

and we will arrive at inequality (1.24).

1.6. The method of least squares. The approximate solution $u_n \in H_n$ of Eqs. (1.1) using the method of least squares is determined from the condition of the minimum of the discrepancy $\|Au_n - f_\delta\|$ in the subspace H_n. This is equivalent to the determination of u_n from the conditions

$$u_n \in H_n,\ \langle Au_n - f_\delta,\ Av_n\rangle = 0\ (\forall v_n \in H_n). \qquad (1.26)$$

Thus, the method of least squares is the projection method (1.2), in which only the spaces $H_n \subset H$ are specified, and $F_n = AH_n$. Obviously, the approximation u_n exists for all n = 1, 2, ...; it is unique if

$$N(A) \cap H_n = 0.$$

Condition (1.16) in this case holds with $\tau = 1,\ n_0 = 1$. From Theorems 1.1 and 1.2 using Lemmas 1.5 and 1.8 we obtain the following result.

Theorem 1.3. Suppose $N(A) = 0,\ f \in R(A),\ \|f_\delta - f\| \leqslant \delta,\ \|u - P_n u\| \to 0$ as $n \to \infty\ (\forall u \in H)$ and for some natural m

$$(\varkappa_n + \varkappa_{n+1})^{1/m} \|(I - P_n)(A^*A)^{1/(2m)}\| \leqslant \text{const}\ (n = 1,\ 2,\ \ldots). \qquad (1.27)$$

If $n = n(\delta)$ in the method of least squares we take (1.26), such that

$$n(\delta) \to \infty,\ \delta\varkappa_{n(\delta)} \to 0 \text{ as } \delta \to 0, \qquad (1.28)$$

then $u_{n(\delta)} \to u_* = A^{-1}f$ as $\delta \to 0$. The same convergence occurs if for n(δ) we take the first of the numbers n = 1, 2, ..., for which $\|Au_n - f_\delta\| \leqslant b\delta\ (b = \text{const} > 1)$.

From Lemmas 1.1 and 1.2 it follows that in the case of the method of least squares we always have $\varkappa_n^* \leqslant \varkappa_n$, and when (1.27) holds we also have $\varkappa_n \leqslant \varkappa_{n}^*/\tau^*$.

1.7. The method of least error. We shall specify the subspaces $F_n \subset F\ (n = 1,\ 2,\ \ldots)$ and shall assume $H_n = A^*F_n\ (n = 1,\ 2,\ \ldots)$. The projection method (1.2) will take the form

$$u_n \in A^*F_n,\ \langle Au_n - f_\delta,\ z_n\rangle = 0\ (\forall z_n \in F_n) \qquad (1.29)$$

(the method of least error).

Theorem 1.4. Suppose $N(A) = 0,\ N(A^*) = 0,\ f \in R(A),\ \|f_\delta - f\| \leqslant \delta$ and

$$\|z - Q_n z\| \to 0 \text{ as } n \to \infty\ (\forall z \in F). \qquad (1.30)$$

Then the method of least error (1.29) for all n = 1, 2, ... determines the unique approximatin u_n. For the a priori choice $n = n(\delta)$ in accordance with (1.11) we have the convergence $u_{n(\delta)} \to u_* = A^{-1}f$ as $\delta \to 0$. If for some natural m

$$(\varkappa_n^*)^{1/m} \|(I - Q_n)(AA^*)^{1/(2m)}\| \leqslant \gamma,\ (\varkappa_{n+1}^*)^{1/m} \|(I - Q_n)(AA^*)^{1/(2m)}\| \leqslant \text{const}\ (n \geqslant 1), \qquad (1.31)$$

then the convergence $u_{n(\delta)} \to u_*$ also occurs as $\delta \to 0$ in the case when for n(δ) the first of the numbers n = 1, 2, ..., is chosen for which

$$\|Au_n - f_\delta\| \leqslant b\delta,\ b > (1 + \gamma^2)^{m/2}.$$

Proof. It is easy to verify that (1.8) follows from (1.30). Condition (1.10) holds when $\tau^* = 1,\ n_0 = 1$. The application of Theorem 1.1 gives the statement of convergence for the a priori choice $n = n(\delta)$. From the first inequality (1.31) using Lemma 1.5 we directly obtain (1.16) when

$$\tau = (1+\gamma^2)^{-m/2}.$$

Since $(I-Q_n')AP_n=0$, then

$$\|(I-Q_n')A\| = \|(I-Q_n')A(I-P_n)\| \leqslant \|(I-P_n)^* A^*\|.$$

In this case P_n projects to A^*F_n; applying the analog of Lemma 1.8, we obtain $\|(I-P_n)A^*\| \leqslant \|(I-Q_n)(AA^*)^{1/(2m)}\|^m$. Therefore from the second condition (1.31) follows (1.18). The application of Theorem 1.2 gives the statement on the convergence when choosing $n=n(\delta)$ using the discrepancy. Theorem 1.4 is proved.

In the case of the method of least errors we always have $\varkappa_n < \varkappa_\tau^*$, and when the first of conditions (1.31) holds we also have $\varkappa_n^* \leqslant (1+\gamma^2)^{m/2}\varkappa_n$.

1.8. Galerkin's method. In the case H = F, $H_n = F_n$ the projection method (1.2) becomes Galerkin's method. The approximate solution of Eq. (1.1) is determined from the conditions

$$u_n \in H_n,\ \langle Au_n - f_\delta,\ v_n\rangle = 0\ (\forall v_n \in H_n). \tag{1.32}$$

Theorem 1.5. Suppose $H=F,\ A=A^*>0,\ f\in R(A),\ \|f_\delta - f\|\leqslant\delta,$

$$\|u-P_n u\| \to 0 \text{ as } n\to\infty\ (\forall u \in H).$$

Suppose for some natural m

$$\varkappa_n^{1/m}\|(I-P_n)A^{1/m}\| \leqslant \gamma,\ \varkappa_{n+1}^{1/m}\|(I-P_n)A^{1/m}\| \leqslant \text{const}\ (n\geqslant 1). \tag{1.33}$$

Then Galerkin's method (1.32) for all n = 1, 2, ... determines the approximate solution u_n. For the a priori choice $n=n(\delta)$ in accordance with condition (1.11) we have the convergence $u_{n(\delta)} \to u_* = A^{-1}f$ as $\delta \to 0$. The same convergence occurs if for $n=n(\delta)$ we take the first of n = 1, 2, ..., for which $\|Au_n - f_\delta\| \leqslant b\delta$, $b=\text{const}$, $b>(1+\gamma^2)^{m/2}$ when $m\geqslant 2$, $b>1+\gamma$ when m = 1.

The proof follows directly from Theorems 1.1, 1.2 and Lemmas 1.7, 1.3.

1.9. Discussion. Of the above projection methods the method of least error (1.29) is particularly interesting. Using the arbitrary extremely dense sequence of subspaces F_n, the method of least error is a regularizing algorithm for all the problems (1.1) with $N(A)=0,\ N(A^*)=0,\ f\in R(A)$, provided we choose $n=n(\delta)$ using conditions (1.11). It should, however, be noted that this choice of n is, in practice, difficult - we need to calculate or estimate $\varkappa_n^* = \sup_{z_n\in F_n}\|z_n\|/\|A^*z_n\|$. Moreover, conditions (1.11) provide only qualitative data about $n=n(\delta)$. In practical problems, although δ is small it is impossible to reduce it the more it approaches zero. The choice $n=n(\delta)$ using the discrepancy is well fitted to practical problems, but has a limited sphere of applicability (see condition 1.31).

Condition (1.31) and the similar conditions - (1.27) and (1.33) - imposed on the projection subspaces are rather burdensome. Suppose operator A is completely continuous, $\lambda_n (\lambda_1 \geqslant \lambda_2 \geqslant \ldots; \lambda_n \to 0$ as $n \to \infty$ are its singular numbers, i.e., the eigenvalues of the operator $(A^*A)^{1/2}$. It is easy to show that for any n-dimensional subspace H_n the following inequalities hold:

$$\|(I-P_n)(A^*A)^{1/(2m)}\| \geqslant \lambda_{n+1}^{1/m},\ \varkappa_n \geqslant \lambda_n^{-1},$$

therefore condition (1.27) is equivalent to the requirements

$$\|(I-P_n)(A^*A)^{1/(2m)}\| \asymp \lambda_{n+1}^{1/m},\ \varkappa_n \asymp \lambda_n^{-1}.$$

Far from all the subspaces H_n have this approximation property and stability property, and the standard subspaces used when constructing numerical methods (the subspaces of polynomials, splines, finite elements) have these properties for far from all the operators A.

In Sect. 2 we isolate a class of integral operators for which condition (1.27) and similar conditions hold when including spline subspaces. It is clear that the kernel of this integral operator cannot be infinitely smooth.

1.10. The case of an approximately specified operator. The results of sections 1.3-1.8 assume a generalization to the case when only the approximation $A_\eta \in \mathcal{L}(H, F), \|A_\eta - A\| \leqslant \eta$ instead of the operator $A \in \mathcal{L}(H, F)$ is in our arrangement. We shall confine ourselves to formulating the analogs of Theorems 1.3-1.5. The corresponding proofs and additional material can be seen in [26,27]. (We will present Theorems 1.3'-1.5' in a stronger formulation than in [26,27]; the strengthening is achieved by including Lemmas 1.5, 1.7, 1.8).

Theorem 1.3'. Suppose $N(A)=0$, $f \in R(A)$, $\|f_\delta - f\| \leqslant \delta$, $\|A_\eta - A\| \leqslant \eta$, $\|u - P_n u\| \to 0$ as $n \to \infty (\forall u \in H)$, and suppose for some natural m

$$(\varkappa_n + \varkappa_{n+1})^{1/m} \|(I-P_n)(A^*A)^{1/(2m)}\| \leqslant \text{const}\ (n = 1, 2, \ldots).$$

Then the method of least squares

$$u_n \in H_n,\ (A_\eta u_n - f_\delta,\ A_\eta v_n) = 0\ (\forall v_n \in H_n)$$

when $\eta\varkappa_n < 1$ determines the unique approximation u_n. If $n = n(\delta, \eta)$ is chosen, such that

$$n(\delta, \eta) \to \infty,\ (\delta + \eta)\varkappa_{n(\delta, \eta)} \to 0 \text{ as } \delta, \eta \to 0, \tag{1.34}$$

then $u_{n(\delta, \eta)} \to u_*$ as $\delta, \eta \to 0$. For fairly small δ, η the choice $n = n(\delta, \eta)$ will be implemented using the rule: $n = n(\delta, \eta)$ is the first of the numbers n = 1, 2, ..., for which

$$\|A_\eta u_n - f_\delta\| \leqslant b(\delta + \|u_n\|\eta),\ b = \text{const} > 1; \tag{1.35}$$

for the choice $u_{n(\delta,\eta)} \to u_*$ as $\delta, \eta \to 0$.

Note. To guarantee the choice $n = n(\delta, \eta)$ using the discrepancy for all δ, η we can refine the rule for choosing n, for example, thus: we shall specify the numbers $b > 1$, $d \in (0, 1)$ and the function g(n), such that $\varkappa_n \leqslant g(n) \leqslant c\varkappa_n$; we shall choose the first n from n = 1, 2, ..., for which $\eta g(n) > d$ or (1.35) holds. For small δ, η this addition does not affect the choice $n = n(\delta, \eta)$. Similar notes concern Theorems 1.4', 1.5' below.

Theorem 1.4'. Suppose

$$N(A) = 0,\ N(A^*) = 0,\ f \in R(A),\ \|f_\delta - f\| \leqslant \delta,\ \|A_\eta - A\| \leqslant \eta$$

and $\|z - Q_n z\| \to 0$ as $n \to \infty$ $(\forall z \in F)$. Then the method of least error

$$u_n \in A_\eta^* F_n,\ \langle A_\eta u_n - f_\delta,\ z_n \rangle = 0\ (\forall z_n \in F_n)$$

when $\eta\varkappa_n^* < 1$ determines the unique approximation u_n. If $n = n(\delta, \eta)$ is chosen such that

$$n(\delta, \eta) \to \infty,\ (\delta + \eta)\varkappa_{n(\delta,\eta)}^* \to 0 \quad \text{as} \quad \delta, \eta \to 0,$$

then $u_{n(\delta,\eta)} \to u_*$ as $\delta, \eta \to 0$. If for some natural m the following conditions hold:

$$(\varkappa_n^*)^{1/m} \|(I - Q_n)(AA^*)^{1/(2m)}\| \leqslant \gamma,\ (\varkappa_{n+1}^*)^{1/m} \|(I - Q_n)(AA^*)^{1/(2m)}\| \leqslant \text{const}\ (n \geqslant 1),$$

and δ, η are fairly small, then $n = n(\delta, \eta)$ is chosen using the rule; $n = n(\delta, \eta)$ is the first of the numbers n = 1, 2, ..., for which (1.35) holds, where b is a fairly large constant; for this choice $u_{n(\delta,\eta)} \to u_*$ as $\delta, \eta \to 0$.

Theorem 1.5'. Suppose $H = F,\ A = A^* > 0,\ A_\eta = A_\eta^* > 0,\ f \in R(A)$,

$$\|f_\delta - f\| \leqslant \delta,\ \|A_\eta - A\| \leqslant \eta,\ \|u - P_n u\| \to 0 \quad \text{as} \quad n \to \infty\ (\forall u \in H).$$

Suppose for some natural m

$$\varkappa_n^{1/m} \|(I - P_n) A^{1/m}\| \leqslant \gamma,\ \varkappa_{n+1}^{1/m} \|(I - P_n) A^{1/m}\| \leqslant \text{const}\ (n \geqslant 1).$$

Then Galerkin's method $u_n \in H_n,\ \langle A_\eta u_n - f_\delta,\ v_n \rangle = 0\ (\forall v_n \in H_n)$ when $\eta\varkappa_n < (1 + \gamma^2)^{-m/2}$ determines the unique approximation u_n. If $n = n(\delta, \eta)$ is chosen using conditions (1.34), then $u_{n(\delta,\eta)} \to u_*$ as $\delta, \eta \to 0$. For fairly small δ, η $n = n(\delta, \eta)$ is chosen using the rule: $n = n(\delta, \eta)$ is the first of the numbers n = 1, 2, ..., for which (1.35) holds where b is a fairly large constant; for this choice $u_{n(\delta,\eta)} \to u_*$ for $\delta, \eta \to 0$.

2. SOLUTION OF INTEGRAL EQUATIONS WITH KERNELS OF THE GREEN FUNCTION TYPE

2.1. Description of the class of equations. Consider the integral equation of the first kind

$$(Au)(t) \equiv \int_0^1 K(t, s)\, u(s)\, ds = f(t) \quad (0 \leqslant t \leqslant 1). \qquad (2.1)$$

We will impose the following conidtions on the kernel K(t, s):

1°) K(t,s) has t continuous partial derivatives up to order m - 2 inclusively in the whole square $0 \leqslant t, s \leqslant 1$, and when t < s and t > s also up to order m - 1 and m, which are extended to the continuous functions in the closed triangles $0 \leqslant t \leqslant s \leqslant 1$ and $0 \leqslant s \leqslant t \leqslant 1$;

2°) $\left.\frac{\partial^{m-1} K(t, s)}{\partial t^{m-1}}\right|_{t=s+0} - \left.\frac{\partial^{m-1} K(t, s)}{\partial t^{m-1}}\right|_{t=s-0} = a(s)$,

where a is a continuous function in [0, 1], and $a(s) \neq 0$ for all $s \in [0, 1]$;

3°) among the 2m functions

$$a_j^0(s) = \left.\frac{\partial^j K(t, s)}{\partial t^j}\right|_{t=0},\ a_j^1(s) = \left.\frac{\partial^j K(t, s)}{\partial t^j}\right|_{t=1}\ (j = 0, 1, \dots, m-1)$$

there are no more than m linearly independent functions in [0, 1]. Here $m \geq 1$ is a natural number.

Volterra's kernel satisfies, for example, conditions 1°)-3°) (then $K(t, s) \equiv 0$ when t < s), if

$$\partial^i K(t, s)/\partial t^i|_{t=s+0} = 0\ (i = 0, \dots, m-2),\ \partial^{m-1} K(t, s)/\partial t^{m-1}|_{t=s+0} \neq 0.$$

Condition 3°) is equivalent to the following: K(t, s) like the function of t for all $s \in (0, 1)$ satisfies some linearly independent boundary value conditions

$$\sum_{j=0}^{m-1} [\alpha_{ij} z^{(j)}(0) + \beta_{ij} z^{(j)}(1)] = 0\ (i = 1, \dots, m). \tag{2.2}$$

Green's function of any differential operator

$$L_m z = \sum_{j=0}^{m} b_j(t) z^{(j)},\ b_j \in C[0, 1],\ b_m(t) = 1/a(t)$$

satisfies conditions 1°)-3°) for any boundary value conditions (2.2), for which the homogeneous equation $L_m z = 0$ has only a zeroth solution. But the class of kernels satisfying conditions 1°)-3°) is not exhausted by Green's functions.

2.2 The domain of integral operator values. The following two statements are proved in an elementary way.

Lemma 2.1. Suppose conditions 1°) and 2°) hold. Then the integral operator A, determined in (2.1), operates and is bounded from $L^2(0, 1)$ to $W^{m,2}(0, 1)$, and is of the Fredholm type between this pair of spaces and $\operatorname{Ind} A = -m$.

Lemma 2.2. Suppose conditions 1°)-3°) hold and suppose the homogeneous integral equation Au = 0 has only a zeroth solution in $L^2(0, 1)$. Then

$$W_0^{m,2}(0, 1) \subset AL^2(0, 1) \subset W^{m,2}(0, 1). \tag{2.3}$$

Here, as usual, $W_0^{m,2} = \{z \in W^{m,2} : z^{(j)}(0) = z^{(j)}(1) = 0,\ j = 0, 1, \dots, m-1\}$. In the conditions of Lemma 2.2 we can give a more accurate description $R(A) = AL^2$ - it consists of the functions $z \in W^{m,2}$, which satisfy the boundary value conditions (2.2). Note the following: if conditions 1°) and 2°) hold, then $\nu = \lambda + \mu - m$, where ν is the codimension of the subspace $W_0^{m,2} \cap AL^2$ in $W_0^{m,2}$, $\lambda = \dim N(A)$, $\mu = \dim \mathfrak{M}$, where $\mathfrak{M}$ is the linear envelope of the functions a_j^0 and a_j^1 $(j = 0, 1, \ldots, m-1)$. In particular, N(A) is nontrivial if $\mu < m$, and when $\mu > m$ the insertion $AL^2 \supset W_0^{m,2}$ is impaired.

The insertions (2.3) have a principal value which is required later.

2.3. The property of the transposed operator. The linear operator B, which operates from L^2 to some space inserted into L^2, can be considered like the operator from L^2. We shall denote the operator adjoint to B^T like the operator in $L^2: \langle Bu, v\rangle_{L^2} = \langle u, B^T v\rangle_{L^2}$ $(\forall u, v \in L^2)$.

Lemma 2.3. Suppose $B \in \mathcal{L}(L^2, W^{m,2})$ is such that $R(B) \supseteq W_0^{m,2}$. Then

$$\|B^T v\|_{L^2} > c_0 \|D^{(-m)} v\|_{L^2} \ (\forall v \in L^2), \ c_0 = \text{const} > 0, \tag{2.4}$$

where $D^{(-m)} v = D^m \Gamma_m v$, $\Gamma = \Gamma_m : L^2 \to W_0^{m,2}$ is inverse to the differential operator D^{2m} for the boundary conditions $z^{(j)}(0) = z^{(j)}(1) = 0$, $j = 0, 1, \ldots, m-1$.

Proof. Introducing in $W_0^{m,2}$ the following scalar product:

$$\langle z_1, z_2\rangle_m = \int_0^1 z_1^{(m)}(t) \bar{z}_2^{(m)}(t)\, dt = \langle D^m z_1, D^m z_2\rangle_0,$$

we have

$$\langle y, z\rangle_0 = \langle \Gamma y, z\rangle_m \ (\forall y \in L^2, z \in W_0^{m,2}).$$

We shall use H to denote the subspace in L^2 which consists of all the functions $u \in L^2$, which are orthogonal to N(B), for which $Bu \in W_0^{m,2}$. From the insertion $R(B) \supseteq W_0^{m,2}$ and the finite codimension $W_0^{m,2}$ in $W^{m,2}$ it follows that H is a closed subspace in L^2. We shall use $C \in \mathcal{L}(H, W_0^{m,2})$ to denote the narrowing of the operator B in H. This is a continuous invertible operator, which converts H into $W_0^{m,2}$. Using Banach's theorem $C^{-1} \in \mathcal{L}(W_0^{m,2}, H)$ is also continuous. Formulating $c_0 = 1/\|C^{-1}\|$, we have

$$\begin{aligned} \|B^T v\|_0 &= \sup_{u \in L^2,\ \|u\|_0 \le 1} \langle B^T v, u\rangle_0 = \sup_{u \in L^2,\ \|u\|_0 \le 1} \langle v, Bu\rangle_0 > \\ &> \sup_{u \in H,\ \|u\|_0 \le 1} \langle v, Cu\rangle_0 = \sup_{u \in H,\ \|u\|_0 \le 1} \langle \Gamma v, Cu\rangle_m = \\ &= \sup_{w \in W_0^{m,2},\ \|C^{-1} w\|_0 \le 1} \langle \Gamma v, w\rangle_m > \sup_{w \in W_0^{m,2},\ \|w\|_m \le c_0} \langle \Gamma v, w\rangle_m = \\ &= c_0 \|\Gamma v\|_m = c_0 \|D^m \Gamma v\|_0 = c_0 \|D^{(-m)} v\|_0 \ (\forall v \in L^2). \end{aligned}$$

2.4. The spline space. Suppose $k > 1$, $0 \le l \le k-1$, $h = 1/n$. We shall use S_{hkl} to denote the set of functions $u_n \in W^{l,2}(0, 1)$, which, in each segment $[(i-1)h, ih]$, $i = 1, \ldots, n$, are polynomials of the power $\le k-1$. We recall some fundamental properties of the splines (see [44]):

1) for any function $u \in W^{l,2}$ $u_n \in S_{hkl}$ $(n=1, 2, \dots)$, exist, such that

$$\|D^i(u_n - u)\|_0 \leq ch^{j-i}\|D^j u\|_0 \quad (0 \leq i < j \leq k,\ i \leq l) \tag{2.5}$$

(the property of approximation);

2) for any $u_n \in S_{hkl}$ the following inequality holds:

$$\|D^j u_n\|_0 \leq c' n^{j-i}\|D^i u_n\|_0 \quad (0 \leq i < j \leq l) \tag{2.6}$$

(the property of stability). The constants c and c' do not depend on n, u and u_n.

2.5. The method of least error. We shall consider the integral operator A in Eq. (2.1) like the operator from H = $L^2(0, 1)$ into F = $L^2(0, 1)$. We will assume

$$F_n = S_{hkl} \quad (k > 1,\ 0 \leq l \leq k-1)$$

and shall apply the method of least error

$$u_n \in A^T S_{hkl},\ \langle Au_n - f_\delta,\ v_n\rangle_0 = 0 \ (\forall v_n \in S_{hkl}), \tag{2.7}$$

where

$$\langle u,\ v\rangle_0 = \int_0^1 u(t)\,\overline{v}(t)\,dt,\quad (A^T v)(s) = \int_0^1 \overline{K(t,\ s)}\,v(t)\,dt.$$

We will estimate the quantity $\varkappa_n^* = \sup\limits_{z_n \in S_{hkl}} \|z_n\|_0/\|A^T z_n\|_0$ (see (1.5)). We will assume that conditions 1°)-3°) of section 2.1 hold and N(A) = 0. Then the insertions (2.3) hold and using Lemma 2.3 $\|A^T z_n\|_0 \geq c_0\|D^{(-m)} z_n\|_0$. The operator $D^{(-m)}$ converts S_{hkl} to $S_{h,k+m,l+m}$, and using the property of spline stability (see inequality (2.6), which we apply to $D^{(-m)}z_n$) we have $\|z_n\|_0 = \|D^m D^{(-m)} z_n\|_0 \leq c' n^m \|D^{(-m)} z_n\|_0$. Finally $\varkappa_n \leq \varkappa_n^* \leq c' c_0^{-1} n^m$ $(n = 1, 2, \dots)$. We will show below that $\|(I - Q_n)(AA^T)^{1/(2m)}\| = O(h)$ and thus conditions (1.31) hold. The proof rests on the following subsidiary result.

Lemma 2.4. If B is a self-adjoint positive operator in L^2 with the domain of values $R(B) \subseteq W^{m,2}$, then $R(B^\alpha) \subseteq W^{\alpha m,2}$ $(0 \leq \alpha \leq 1)$.

Proof. We will take any operator Λ which is self-adjoint and positive definite in L^2 with the area of definition $D(\Lambda) = W^{m,2}$ (such as, for example, $\Lambda = M^{1/2}$, $Mu = (-1)^m u^{(2m)} + u$ with

$$D(M) = \{u \in W^{2m,2} : u^{(j)}(0) = u^{(j)}(1) = 0,\ j = m,\ \dots,\ 2m-1\}).$$

Then (see [45]) $D(\Lambda^\alpha) = W^{\alpha m,2}$ $(0 \leq \alpha \leq 1)$. We will formulate C = B^{-1}. This is also a self-adjoint positive definite operator, and using the condition

$$D(C) = R(B) \subseteq W^{m,2} = D(\Lambda).$$

Using Heinz's theorem (see [46]): $D(C^\alpha) \subseteq D(\Lambda^\alpha)$ $(0 \leq \alpha \leq 1)$, i.e., $R(B^\alpha) \subseteq W^{\alpha m,2}$, which it is also required to prove.

Since $R(A)=R((AA^T)^{1/2})$, then in accordance with (2.3) we have

$$R((AA^T)^{1/2})\subset W^{m,2}.$$

Using Lemma 2.4 it hence follows that $R((AA^T)^{1/(2m)})\subseteq W^{1,2}$; obviously, $(AA^T)^{1/(2m)}$ is closed like the operator from L^2 into $W^{1,2}$, and therefore also bounded. Using (2.5) with i = 0, j = 1, we obtain

$$\|(I-Q_n)(AA^T)^{1/(2m)}\|_{L^2\to L^2}\leqslant c\|D(AA^T)^{1/(2m)}\|_{L^2\to L^2}h.$$

Thus condition (1.31) holds, and

$$(\varkappa_n^*)^{1/m}\|(I-Q_n)(AA^T)^{1/(2m)}\|\leqslant c(c'c_0^{-1})^{1/m}\|D(AA^T)^{1/(2m)}\|\equiv\tau. \qquad (2.8)$$

From Theorem 1.4 we obtain the following result.

Theorem 2.1. Suppose conditions 1°)-3°) of section 2.1 hold, $N(A)=0$, $f\in W^{m,2}$ satisfies the boundary conditions (2.2), and $\|f_\delta-f\|_0\leqslant\delta$. Then Eq. (2.1) has the unique solution $u_*\in L^2$, and the method of least error (2.7) for all n ≥ 1 determines the unique approximation u_n. Coordinating n with δ, such that

$$n(\delta)\to\infty,\ \delta[n(\delta)]^m\to 0 \quad \text{as} \quad \delta\to 0, \qquad (2.9)$$

we have the convergence $\|u_{n(\delta)}-u_*\|_0\to 0$ as δ → 0. The same convergence occurs if for n(δ) we take the first of n = 1, 2, ... (or the first of $n=2^\nu$, $\nu=1, 2, \ldots$), for which $\|Au_n-f_\delta\|_0\leqslant b\delta$, where $b>(1+\tau^2)^{m/2}$ (see (2.8)).

In comparison with the formulation of Theorem 1.4 we omitted the condition $N(A^T)=0$, which follows from the density R(A) in L^2 (see (2.3)).

2.6. The method of least squares. We will assume $H_n=S_{hkl}(k\geqslant 1, 0\leqslant l\leqslant k-1)$ and shall use the method of least squares to solve Eq. (2.1):

$$u_n\in S_{hkl},\ (Au_n-f_\delta,\ Av_n)_0=0\ (\forall v_n\in S_{hkl}). \qquad (2.10)$$

To estimate the quantity $\varkappa_n=\sup\limits_{u_n\in S_{hkl}}\|u_n\|_0/\|Au_n\|_0$ (see (1.4)) we can use the ready result of the previous section, changing the roles of A and A^T. At the same time we should impose on K(t, s) the conditions obtained from conditions 1°)-3°) of section 2.1, by changing the roles of t and s. As a result we have the estimate

$$\varkappa_n^*\leqslant\varkappa_n\leqslant\text{const}\cdot n^m\ (n=1, 2, \ldots).$$

In addition, $\|(I-P_n)(A^TA)^{1/(2m)}\|\leqslant\text{const}\cdot h$, therefore (1.27) holds and from Theorem 1.3 we obtain the following result.

Theorem 2.2. Suppose the kernel K(t, s) satisfies the analogs of conditions 1°)-3°) of section 2.1, which are obtained by changing the roles of t and s, and suppose $N(A^T)=0$. Suppose $f\in R(A)$, $\|f_\delta-f\|_0\leqslant\delta$. Then Eq. (2.1) has the unique solution $u_*\in L^2$, and the method of least squares (2.10) when n ≥ 1 deter mines the unique approximation u_n. Coordinating n with δ in accordance with (2.9), we have the convergence $\|u_{n(\delta)} - u_*\|_0 \to 0$ as $\delta \to 0$. The same convergence occurs if for n(δ) we take the first of n = 1, 2, ... (or the first of $n = 2^\nu$, $\nu = 1, 2, \ldots$), for which $\|Au_n - f_\delta\|_0 \leqslant b\delta$, $b = \text{const} > 1$.

2.7. Galerkin's method. We will assume $H_n = F_n = S_{hkl}$ $(k \geqslant 1,\ 0 \leqslant l \leqslant k-1)$ and will use Galerkin's method to solve Eq. (2.1):

$$u_n \in S_{hkl} \quad \langle Au_n - f_\delta,\ v_n\rangle_0 = 0 \quad (\forall v_n \in S_{hkl}). \tag{2.11}$$

Theorem 2.3. Suppose the kernel K(t, s) is symmetric and determines the operator $(A = A^* > 0)$, which is positive in L^2. Suppose conditions 1°)-3°) of section 2.1 hold, and suppose $f \in W^{m,2}$ satisfies the boundary conditions (2.2), $\|f_\delta - f\|_0 \leqslant \delta$. Then Eq. (2.1) has the unique solution $u_* \in L^2$, and Galerkin's method (2.11) for all n ≥ 1 determines the unique approximation u_n. Coordinating n with δ using condition (2.9), we have the convergence $\|u_{n(\delta)} - u_*\|_0 \to 0$ as $\delta \to 0$. The same convergence occurs if for n(δ) we take the first of the numbers n = 1, 2, ... (or the first of the numbers $n = 2^\nu$, $\nu = 1, 2, \ldots$), for which $\|Au_n - f_\delta\|_0 \leqslant b\delta$, where $b > (1 + \gamma^2)^{m/2}$ ($b > 1 + \gamma$ in the case m = 1), and γ is a constant from inequality (2.12).

The proof is based on Theorem 1.5 and Lemma 2.4. As in the case of Theorem 2.1, we have $\varkappa_n \leqslant c' c_0^{-1} n^m$, therefore

$$\varkappa_n^{1/m} \|(I - P_n) A^{1/m}\| \leqslant \gamma = (c' c_0^{-1})^{1/m} c \|DA^{1/m}\|_{L^2 \to L^2}, \tag{2.12}$$

which guarantees the satisfaction of condition (1.33).

2.8. Other applications. Insertion (2.3) proved to be conclusive for applying the theorems of the previous paragraph. There exist a number of other problems for which we manage to establish similar insertions.

It is shown in [47] that the insertion (2.3) holds for the integral equation (2.1) with the difference kernel $K(t, s) = K(t - s)$ if $K(x)$ is determined for all $x \in (-\infty, \infty)$ and its Fourier transformation is

$$\int_{-\infty}^{\infty} K(\tau) e^{-ix\tau} d\tau = c\,[(x - x_1)(x - x_2) \ldots (x - x_m)]^{-1},$$

where c is a constant, and x_j, $j = 1, \ldots, l$, are complex numbers with nonzero imaginary parts.

Operators A of the integral equations

$$\int_\Gamma \log|t - s|\, u(s)\, ds = f(t), \quad \int_\Gamma \log[\operatorname{dist}(s,\ t)]^{-1} u(s)\, ds = f(t),$$

where $\Gamma \in C^\infty$ is a closed contour in R^2, such that $|t - s| < 1$ when $t, s \in \Gamma$, have the property $AL^2(\Gamma) = W^{1,2}(\Gamma)$ (see [17,18,24]).

For an equation with Radon's transform operator

$$(Au)(t_1, t_2) \equiv \int_{t_1 s = t_1} u(s)\,ds = f(t_1, t_2), \; A \in \mathcal{L}(L^2(\Omega_1), L^2(\Omega_2)),$$

where Ω_1 is the bounded domain in R^2, $\Omega_2 = \{(t_1, t_2): t_1 \in R^1, t_2 \in R^2, |t_2| = 1\}$ is a unit cylinder in R^3, the property $\|Au\|_{L_2(\Omega_2)} \asymp \|u\|_{(W^{1,2,2}(\Omega_1))^*}$ is known (see [38]).

REFERENCES

1. A. N. Tikhonov and V. Ya. Arsenin, Methods of Solving Ill-Posed Problems [in Russian], Second Edition, Moscow, 1979.
2. V. K. Ivanov, V. V. Vasin, and V. P. Tanana, The Theory of Linear Ill-Posed Problems and its Applications [in Russian], Moscow, 1978.
3. M. M. Lavrent'ev, Some Ill-Posed Problems of Mathematical Physics [in Russian], Novosibirsk, 1962.
4. V. A. Morozov, Methods of Solving Unstable Problems [in Russian], Moscow, 1967.
5. O. A. Liskovets, Variation Methods of Solving Unstable Problems [in Russian], Minsk, 1981.
6. O. A. Liskovets, "The theory and methods of solving ill-posed problems," Itogi Nauki i Tekhniki. Matem. Analiz, vol. 20, pp. 116-178, 1982.
7. G. M. Vainikko, Methods of Solving Linear Ill-Posed Problems in Hilbert Spaces [in Russian], Tartu, 1982.
8. A. N. Tikhonov and V. I. Dmitriev, "A method of calculating the distribution of a current in a set of linear vibrators and diagrams of the directivity of this system," Vychisl. Metody i Programmirov, no. 10, pp. 3-8, 1968.
9. A. S. Apartsin and A. B. Bakushinskii, "An approximate solution of integral Volterra equations of the first kind using the method of quadrature sums," Differents. i Integral'n. Uravneniya, no. 1, pp. 248-258, Irkutsk, 1973.
10. A. S. Apartsin, Discretization Methods of Regularizing Integral Volterra Equations of the First Kind [in Russian], Kandid. Dissert., Novosibirsk, 1983.
11. C. T. H. Baker, The Numerical Treatment of Integral Equations, Oxford, 1977.
12. H. Brunner, "A survey of recent advances in the numerical treatment of Volterra integral and integro-differential equations," J. Comput. and Appl. Math., vol. 8, no. 3, pp. 213-229, 1982.
13. V. V. Ivanov and V. U. Kudrinskii, "An approximate solution of linear operator equations in a Hilbert space using the method of least squares, I," Zh. Vychisl. Mat. i Mat. Fiz., vol. 6, no. 5, pp. 831-841, 1966.
14. V. N. Strakhov, "One method of approximately solving linear ill-posed problems," Zh. Vychisl. Mat. i Mat. Fiz., vol. 10, no. 1, pp. 204-210, 1970.
15. A. B. Bakushinskii, "Observations on the Kupradze-Aleksidze method," Differents. Uravneniya, vol. VI, no. 7, pp. 1298-1301, 1970.
16. F. Natterer, "Regularisierung schlecht gestellter Probleme durch Projektionsverfahren," Numer. Math., vol. 28, no. 3, pp. 329-341, 1977.
17. G. C. Hsiao and W. L. Wendland, "A finite element method for some integral equations of the first kind," J. Math. Anal. and Appl., vol. 58, no. 3, pp. 449-481, 1977.
18. G. R. Richter, "Numerical solution of integral equations of the first kind with nonsmooth kernels," SIAM J. Numer. Anal., vol. 15, no. 3, pp. 511-522, 1978.
19. T. J. Seidman, "Convergent approximation methods for ill-posed problems," Contr. and Cybern., vol. 10, no. 1-2, pp. 31-71, 1981.
20. G. C. Hsiao and W. L. Wendland, "The Aubin-Nitsche lemma for integral equations," J. Integr. Equat., vol. 3, no. 4, pp. 299-315, 1981.
21. D. N. Arnold and W. L. Wendland, "On the asymptotic convergence of co-

location methods," Math. Comput., vol. 41, no. 164, pp. 349-381, 1983.

22. F. Natterer, "On the order of regularization methods," in: Ill-Posed Probl. and Numer. treat., G. Hammerlin and K.-H. Hoffman (Editors), pp. 189-203, Birkhauser, Basel-Boston, Mass, 1983.

23. G. C. Hsiao, "The finite element method for a class of ill-posed integral equations," in: Ill-Posed Probl. and Numer. treat., G. Hammerlin and K.-H. Hoffmann (Editors), pp. 117-131, Birkhauser, Basel-Boston, Mass., 1983.

24. W. L. Wendland, "Boundary element methods and their asymptotic convergence," in: Theoretical Acoustics and Numerical Techniques, P. Fillipi (Editor), pp. 135-216, Vienna, New YOrk, 1983.

25. H. W. Engl, "On the convergence of regularization methods for ill-posed linear operator equations," in: Ill-Posed Probl. and Numer. treat. Conf., Oberwolffach, 26 September-2 October 1982, pp. 81-95, Basel et al., 1983.

26. U. A. Khyamarik, "Projection methods for regularizing ill-posed problems," Tr. Vychisl. Tsentra Tartusk. Un-ta, no. 50, pp. 69-90, 1983.

27. U. A. Khyamarik, "The disrepancy principle for selecting dimensions when solving ill-posed problems using projection methods," Uchen. Zap. Tartusk. Un-ta, no. 672, pp. 27-34, 1984.

28. U. A. Khyamarik, "Reularized projection methods for ill-posed problems," Izv. AN ESSR. Fiz.-Matem., vol. 33, no. 3, pp. 266-276, 1984.

29. N. Z. Nashed and G. Wahba, "Convergence rates of approximate least squares solutions of linear integral and operator equations of the first kind," Math. Comput., vol. 28, no. 125, pp. 69-80, 1974.

30. V. A. Zheludev, "The correctness of one class of equation in contractions," Zh. Vychisl. Mat. i Mat. Fiz., vol. 14, no. 3, pp. 610-630, 1974.

31. A. L. Gaponenko and U. L. Gaponenko, "One regularization method for operator equations of the first kind," Zh. Vychisl. Mat. i Mat. Fiz., vol. 16, no. 3, pp. 577-584, 1976.

32. M. V. Bushmanova, "Finite-dimensional approximations to the solution of a linear operator equation of the first kind," Izv. VUZ. Mat. [Soviet Mathematics (IZV. VUZ.)], no. 9, pp. 11-17, 1977.

33. V. V. Rononin and V. A. Tsetsokho, "A numerical solution of an integral equation of the first kind with a logarithmic peculiarity using the interpolation and collocation method," Zh. Vychisl. Mat. i Mat. Fiz., vol. 21, no. 1, pp. 40-53, 1981.

34. A. M. Denisov, "A method of solving equations of the first kind in a Hilbert space," DAN SSSR, vol. 274, no. 3, pp. 528-530, 1984.

35. U. L. Gaponenko, "The power of resolution and accuracy of solution of an ill-posed problem for a fixed level of error," Zh. Vychisl. Mat. i Mat. Fiz., vol. 24, no. 4, pp. 483-490, 1984.

36. E. I. Sakalauskas, "The Galerkin-Tikhonov regularization method in the problem of numerical differentiation," Zh. Vychisl. Mat. i Mat. Fiz., vol. 24, no. 11, pp. 1742-1747, 1984.

37. V. K. Gorbunov, "Reduction of linear integral equations with a uniform error in the right-hand side," Zh. Vychisl. Mat. i Mat. Fiz., vol. 25, no. 2, pp. 210-223, 1985.

38. F. Natterer, "The finite element method for ill-posed problems," RAIRO, Anal. Numer., vol. 11, no. 3, pp. 271-278, 1977.

39. V. V. Vasin, "Discrete convergence and finite-dimensional approximation of regularizing algorithms," Zh. Vychisl. Mat. i Mat. Fiz., vol. 19, no. 1, pp. 11-21, 1979.

40. A. R. Danilin and V. P. Tanana, "Necessary and sufficient conditions of convergence of approximations of linear ill-posed problems in a Hilbert space. Zh. Vuchisl. Mat. i Mat. Fiz., vol. 24, no. 5, pp. 633-639, 1984.

41. G. M. Vainikko, "Projection methods in ill-posed problems," in: Variatsion.-Rasnostn. Metody v Matem. Fiz., pp. 42-50, Moscow, 1984.

42. T. Kato, Perturbation Theory of Linear Operators [Russian translation], Moscow, 1972.

43. M. A. Krasnosel'skii, G. M. Vainikko, and P. P. Zabreiko, et al., An

Approximate Solution of Operator Equations [in Russian], Moscow, 1969.
44. J.-P. Auben, An Approximate Solution of Elliptical Boundary Value Problems [Russian translation], Moscow, 1977.
45. J.-L. Lions and E. Majenes, Nonhomogeneous Boundary Value Problems and Their Applications [Russian translation], Moscow, 1971.
46. M. A. Krasnosel'skii, P. P. Zabreiko, E. M. Pustyl'nik et al., Integral Operators in Spaces of Summed Functions [in Russian], Moscow, 1966.
47. B. V. Pal'tsev, "The normal solvability of ill-posed integral equations of the first kind in a segment," Sib. Matem. Zhurn., vol. XVIII, no. 1, pp. 195-211, 1977.

18 April 1985 Tartu