ON THE OPTIMALITY OF REGULARIZATION METHODS

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I. INTRODUCTION

Let M be a given set which contains an exact solution of an ill-posed problem. A method to solve this problem is called optimal on M if the maximal error of the method, compared with other methods, is minimal when the exact solution varies in M and the data vary in their given neighbourhoods. Here the concept of maximal error yet needs to be made more precise. For equations with polluted right-hand term this concept is rather natural and well known (Section II). If the operator is polluted also, there are different possibilities to define this concept, and we have chosen the one (Section III) which has the following property: if one has an optimal method in case with polluted right-hand term then it is easy to construct an optimal method in case when the operator is polluted also (Section IV). After that we discuss the optimality of Tikhonov, iteration and other methods (Section V-X).

The optimality of regularization methods is investigated by many authors, see [1-22] and the references in those works. This lecture is based on [15-20].

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II. OPTIMALITY DEFINITIONS: POLLUTED RIGHT-HAND TERM

Consider an equation

$$Au = f \tag{1}$$

where $A : E \to F$ is an operator (a mapping) from a Banach space E into another Banach space F. We set no assumptions about existence or continuity of the inverse mapping $A^{-1}: F \to E$, and in general problem (1) is ill-posed. In this section we assume that only the right-hand term is polluted: instead of f we have at our disposal an element $f_\delta \in F$, $\|f_\delta - f\| \le \delta$. Any mapping $P : F \to E$ can be treated as a method to solve equation (1), approximate solution is given by $Pf_\delta \in E$. Let $M \subset E$ be a given set. Define the maximal error [7] of method P on M:

$$\varphi(\delta,P) = \varphi(\delta,P,A,M) = \sup_{\substack{u \in M,\ f_\delta \in F \\ \|Au - f_\delta\| \le \delta}} \|Pf_\delta - u\|.$$

A parameter dependent method $P_\delta : F \to E$ is called [7, 20]

- optimal on M if $\varphi(\delta, P_\delta) = \inf_P \varphi(\delta,P)$;
- asymptotically optimal on M if

$$\lim_{\delta \to 0} \left[\varphi(\delta,P_\delta) \,/\, \inf_P \varphi(\delta,P)\right] = 1;$$

- quasioptimal (or order optimal) on M if

$$\varphi(\delta,P_\delta) \le c \inf_P \varphi(\delta,P), \quad 0 < \delta \le \delta_0 \tag{2}$$

(the infimum is taken over all methods $P : F \to E$).

Let us recall an elementary result concerning the behaviour of quantity $\inf_P \varphi(\delta,P)$. For a convex, centrally symmetric set $M \subset E$ and linear operator $A : E \to F$, we have [7, 20]

$$\omega(\delta,A,M) \le \inf_P \varphi(\delta,P,A,M) \le 2\,\omega(\delta,A,M) \tag{3}$$

where

$$\omega(\delta,A,M) = \sup_{u \in M,\ \|Au\| \le \delta} \|u\|.$$

If $M \subset E$ is compact and $A : E \to F$ linear whereby $M \cap \mathcal{N}(A) = \{0\}$ then $\omega(\delta,A,M) \to 0$ as $\delta \to 0$. Here $\mathcal{N}(A) = \{u \in E : Au = 0\}$.

III. OPTIMALITY DEFINITIONS: OPERATOR POLLUTED ALSO

Now we consider the case when the operator A in equation (1) is polluted also: instead of f and A we have at our disposal an element $f_\delta \in F$ and an operator $A_\eta : E \to F$ such that $\|f_\delta - f\| \le \delta$ and

$$\|A_\eta v - Av\| \le \mu\eta \quad \forall v \in M, \quad \mu = \mu_M = \text{const}. \tag{4}$$

Let $\mathfrak{A}$ be a subset of the set of all mappings from E into F. Now any mapping $Q : F \times \mathfrak{A} \to E$ can be treated as a method to solve equation (1), approximate solution $Q(f_\delta, A_\eta) \in E$ now depends on f_δ and A_η. Let us define the maximal error of method Q on M via the formula [19, 20]

$$\psi(\delta,\eta,Q) = \psi(\delta,\eta,Q,A,M) = \sup_{\substack{u \in M,\ f_\delta \in F,\ A_\eta \in \mathfrak{A} \\ \|A_\eta v - Av\| \le \mu\eta\ \forall v \in M \\ \|A_\eta u - f_\delta\| \le \delta + \mu\eta}} \|Q(f_\delta, A_\eta) - u\|.$$

Having this expression for the maximal error we define the optimality concepts in a similar way as in Section II. Namely, a method $Q_{\delta\eta} : F \times \mathfrak{A} \to E$ is called [19, 20]

- $\mathfrak{A}$-optimal on M if $\psi(\delta,\eta,Q_{\delta\eta}) = \inf_Q \psi(\delta,\eta,Q)$;
- asymptotically $\mathfrak{A}$-optimal on M if

$$\lim_{\delta,\eta \to 0} \left[\psi(\delta,\eta,Q_{\delta\eta}) \,/\, \inf_Q \psi(\delta,\eta,Q)\right] = 1;$$

- $\mathfrak{A}$-quasioptimal (or order $\mathfrak{A}$-optimal) on M if

$$\psi(\delta,\eta,Q_{\delta\eta}) \le c \inf_Q \psi(\delta,\eta,Q), \quad 0<\delta\le\delta_0,\ 0<\eta\le\eta_0 \tag{5}$$

(the infimum is taken over all methods $Q : F\times\mathfrak{A} \to E$). Constant $c \geq 1$ in (2) and (5) is called quasioptimality constant; value $c = 1$ corresponds to optimal ($\mathfrak{A}$-optimal) methods.

In case of convex, centrally symmetric set $M\subset E$ and linear operators $A, A_\eta : E \to F$, instead of (3), we have [19]

$$\omega(\delta+\mu\eta,A,M) \le \inf_Q \psi(\delta,\eta,Q,A,M) \le 2\omega(\delta+2\mu\eta,A,M).$$

Another definitions of maximal error of a method $Q : F\times\mathfrak{A} \to E$ are introduced and used in [7, 11].

IV. CONSTRUCTION OF $\mathfrak{A}$-OPTIMAL METHODS

Proposition 1 [19]. Assume that, for any fixed $A\in\mathfrak{A}$, we have at our disposal an optimal ($c = 1$) or a quasioptimal method $P_\delta = Q_\delta(\cdot,A) : F \to E$ on M with a common quasioptimality constant c :

$$\sup_{\substack{u\in M,\ f_\delta\in F\\ \|Au-f_\delta\|\le\delta}} \|Q_\delta(f_\delta,A) - u\| \le c \inf_P \sup_{\substack{u\in M,\ f_\delta\in F\\ \|Au-f_\delta\|\le\delta}} \|Pf_\delta - u\|. \tag{6}$$

Then method $Q_{\delta+\mu\eta} : F\times\mathfrak{A} \to E$ is $\mathfrak{A}$-optimal ($c = 1$), respectively, $\mathfrak{A}$-quasioptimal on M with the same quasioptimality constant:

$$\psi(\delta,\eta,Q_{\delta+\mu\eta},A,M) \le c \inf_Q \psi(\delta,\eta,Q,A,M). \tag{7}$$

Proof. Write (6) for $A_\eta\in\mathfrak{A}$ on error level $\delta+\mu\eta$:

$$\sup_{u\in M,f_\delta\in F,\|A_\eta u-f_\delta\|\le\delta+\mu\eta} \|Q_{\delta+\mu\eta}(f_\delta,A_\eta) - u\|$$

$$\le c \inf_Q \sup_{u\in M,f_\delta\in F,\|A_\eta u-f_\delta\|\le\delta+\mu\eta} \|Q(f_\delta,A_\eta) - u\|.$$

Now take the supremum over $A_\eta\in\mathfrak{A}$ satisfying (4) and apply an inequality of the kind $\sup_x \inf_y g(x,y) \le \inf_y \sup_x g(x,y)$. The result is (7).

V. QUASIOPTIMALITY CONDITIONS

Let E and F be Banach spaces, as above, but A and A_η linear and bounded: $A, A_\eta\in\mathcal{L}(E,F)$.

Proposition 2. Let $M\subset E$ be convex and centrally symmetric. Assume that a method $P_\delta : F\to E$ has the following property: for any $f_\delta\in F$ such that the set $\{u\in M: \|Au - f_\delta\|\le\delta\}$ is non-empty, we have

$$P_\delta f_\delta \in c_1 M, \quad \|A(P_\delta f_\delta) - f_\delta\| \le c_2\delta.$$

Then method P_δ is quasioptimal on M:

$$\varphi(\delta,P_\delta,A,M) \le c \inf_P \varphi(\delta,P,A,M), \quad c = 1+\max\{c_1,c_2\}.$$

Proposition 3. Let again M be convex and centrally symmetric. Assume that a method $Q_{\delta\eta} : F\times\mathfrak{A}\to E$ has the following property: for any $A_\eta\in\mathfrak{A}$ satisfying (4) and any $f_\delta\in F$ such that set $\{u\in M : \|A_\eta u - f_\delta\|\le\delta+\mu\eta\}$ is non-empty, we have

$$u_{\delta\eta} \equiv Q_{\delta\eta}(f_\delta,A_\eta) \in c_1 M, \quad \|A_\eta u_{\delta\eta} - f_\delta\| \le c_2(\delta+\mu\eta).$$

Then method $Q_{\delta\eta}$ is $\mathfrak{A}$-quasioptimal on M:

$$\psi(\delta,\eta,Q_{\delta\eta},A,M) \le c \inf_Q \psi(\delta,\eta,Q,A,M), \quad c = 1+\max\{c_1,c_2\}.$$

These two propositions are easy consequences from the definitions. For details of proofs, see [19, 20].

VI. OPTIMALITY OF TIKHONOV METHOD

Now let E and F be Hilbert spaces, $A \in \mathcal{L}(E,F)$ a linear bounded operator between them. Let H be a third Hilbert space, $H \subset E$ densely and compactly. We search for an optimal method to solve equation (1) on the set

$$M_\rho = \{u \in H : \|u\|_H \le \rho\} \subset E, \quad \rho > 0.$$

Denote by $I_{HE} \in \mathcal{L}(H,E)$ the imbedding operator ($I_{HE} v = v \in E$ for any $v \in H$), and by $J \in \mathcal{L}(E,H)$ its adjoint:

$$(u,v)_E = (Ju,v)_H \quad \forall\, u \in E,\ v \in H.$$

Note that $J \in \mathcal{L}(E,H)$ is injective and compact; considered as an operator from E into E or from H into H, it is compact, self-adjoint and strictly positive. Operator $A \in \mathcal{L}(E,F)$ can also be treated as an operator from H into F. By $A^T \in \mathcal{L}(F,E)$ and $A^* = JA^T \in \mathcal{L}(F,H)$ we denote the corresponding adjoints:

$$(Au,v)_F = (u,A^Tv)_E \quad \forall\, u,v \in E,$$
$$(Au,v)_F = (u,A^*v)_H \quad \forall\, u \in E, v \in H.$$

According to Tikhonov method, we search for a minimizer of functional

$$\|Au - f_\delta\|_F^2 + \alpha\|u\|_H^2, \quad u \in H,$$

where $\alpha > 0$ is the regularization parameter. The minimum is attained at

$$u_\alpha = (\alpha I + A^*A)^{-1}A^*f_\delta = (\alpha L + A^TA)^{-1}A^Tf_\delta$$

where I is the identity operator in H and $L = J^{-1}$ (here L is treated as an unbounded linear operator in E). From a nice result by Melkman and Micchelli [12] it follows that there exists an appropriate parameter choice $\alpha = \alpha(\delta) = \alpha(\delta, A, M_\rho)$ such that Tikhonov method will be optimal on M_ρ:

$$\sup_{\substack{f_\delta \in F, u \in M_\rho \\ \|Au - f_\delta\| \le \delta}} \|u_{\alpha(\delta)} - u\|_E = \inf_P \sup_{\substack{f_\delta \in F, u \in M_\rho \\ \|Au - f_\delta\| \le \delta}} \|Pf_\delta - u\|_E = [\lambda(t_*)]^{1/2}$$

($\lambda(t_*)$ is defined below). For parameter choice $\alpha = \alpha(\delta, A, M_\rho)$, the following prescription holds:

(i) find the greatest eigenvalue $\lambda = \lambda(t)$ of problem

$$Ju = \lambda\left(\frac{t}{\rho^2} I + \frac{1-t}{\delta^2} A^*A\right)u, \quad 0 \le t \le 1;$$

(ii) find $t_* \in [0,1]$ such that $\lambda(t_*) = \min_{0 \le t \le 1} \lambda(t)$;

(iii) put $\alpha = \frac{t_*}{1 - t_*}(\delta/\rho)^2$.

Note that $\lambda(t)$ is a convex functions on $[0,1]$ and it attains its minimum $t_* = t_*(\delta, \rho, A)$; values $t_* = 0$ and $t_* = 1$ (with corresponding $\alpha = 0$, $u_\alpha = A^+f_\delta$ and $\alpha = \infty$, $u_\alpha = 0$) occur only in some non-typical extremal cases. It is easy to construct iterative realizations of this prescription. Simultaneously with optimal α one can calculate the error $[\lambda(t_*)]^{1/2}$ of Tikhonov method (and any other optimal method) on M_ρ.

Consider now the case with polluted operator:

$$A, A_\eta \in \mathcal{L}(E,F), \quad \|A_\eta - A\|_{\mathcal{L}(H,F)} \le \eta.$$

Then, for M_ρ, (4) is fulfilled with $\mu = \rho$. Let $\mathfrak{A} = \mathcal{L}(E,F)$. From Proposition 1 and the result of this Section it follows that there exists a parameter choice $\alpha = \alpha(\delta, \eta, A_\eta, M_\rho)$ such that the Tikhonov method

$$u_\alpha = (\alpha I + A_\eta^* A_\eta)^{-1} A_\eta^* f_\delta = (\alpha L + A_\eta^T A_\eta)^{-1} A_\eta^T f_\delta$$

is $\mathcal{L}(E,F)$-optimal on M_ρ. The prescription for the parameter choice is similar to the one above:

(i') find greatest eigenvalue $\lambda = \lambda(t)$ of problem

$$Ju = \lambda\left[\frac{t}{\rho^2} I + \frac{1-t}{(\delta+\rho\eta)^2} A_\eta^* A_\eta\right]u, \qquad 0 \leq t \leq 1;$$

(ii') find t_* such that $\lambda(t_*) = \min_{0 \leq t \leq 1} \lambda(t)$;

(iii') put $\alpha = \frac{t_*}{1-t_*}\left(\frac{\delta+\rho\eta}{\rho}\right)^2$.

As an example consider an integral equation of the first kind

$$(Au)(t) \equiv \int_a^b K(t,s)u(s)ds = f(t) \qquad (a \leq t \leq b)$$

or any other equation such that the space choice

$$E = L^2(a,b), \quad F = L^2(a,b) \quad \text{with} \quad A \in \mathcal{L}(E,F)$$

is suitable; let

$$H = H^m(a,b), \quad (u,v)_{H^m} = \int_a^b (uv + u^{(m)}v^{(m)})dt, \quad m \geq 1.$$

Then

$$M_\rho = M_\rho^{(m)} = \{u \in H^m(a,b) : \|u\|_{H^m} \leq \rho\};$$

operator $J \in \mathcal{L}(E,H)$ has the form $J = L^{-1}$ where

$$Lu = (-1)^m u^{(2m)} + u, \quad u^{(k)}(a) = u^{(k)}(b) = 0, \ k = m,\dots,2m-1.$$

Tikhonov method $(\alpha I + A^*A)u = A^* f_\delta$ takes the form

$$(\alpha L + A^T A)u = A^T f_\delta, \quad u^{(k)}(a) = u^{(k)}(b) = 0, \quad k = m,\dots,2m-1,$$

where

$$(A^T z)(s) = \int_a^b K(t,s)z(t)dt.$$

Choosing $\alpha = \alpha(\delta, A, M_\rho)$ according to prescription (i)-(iii) we obtain an optimal method on $M_\rho^{(m)}$ when the error is measured in L^2-norm.

For a concrete practical problem, parameter choice (i)-(iii) needs a priori information that the exact solution u belongs to M_ρ with a given ρ. If we only know that u belongs H but we do not know ρ such that $u \in M_\rho$ then the discrepancy principle for the parameter choice is preferable (see Section VII).

VII. QUASIOPTIMAL CHOICES OF PARAMETER IN TIKHONOV METHOD

Let H and F again be Hilbert spaces, but E a Banach space, $H \subset E$ continuously. Consider equation (1), with $A \in \mathcal{L}(H,F)$, and Tikhonov method $u_\alpha = (\alpha I + A^*A)^{-1}A^* f_\delta$ where $A^* \in \mathcal{L}(F,H)$ is the adjoint to $A \in \mathcal{L}(H,F)$. We search for quasioptimal methods on the set

$$M_\rho = \{u \in H : \|u\|_H \leq \rho\} \subset E$$

when the error of the approximate solution is measured by the norm of E, as in Section VI. So far as E is not a Hilbert space, it is not known whether one can obtain an optimal method on the basis of Tikhonov method. On the other hand, there are several possibilities to obtain a quasioptimal method on M_ρ:

1. if both δ and ρ are given, choose $\alpha = (\delta/\rho)^2$;
2. if δ is given (and ρ possibly unknown), choose $\alpha = \alpha(\delta)$ such that $\|Au_\alpha - f_\delta\| = \delta$ (discrepancy principle);
3. if ρ is given (and δ possibly unknown), choose

$\alpha = \alpha(\delta)$ such that $\|u_\alpha\|_H = \rho$.

The quasioptimality constant on M_ρ takes in cases 1, 2 and 3, respectively, values $c = 2.5$, $c = 2$ and $c = 2$. These values do not depend on space E. Actually, having H, F and $A \in \mathcal{L}(H,F)$, we obtain quasioptimal methods in all E-norms simultaneously, if only $H \subset E$ continuously. To be sure that $\omega(\delta, A, M_\rho) \to 0$ as $\delta \to 0$ (see (3)), here some supplementary assumptions must be done, e.g. $H \subset E$ compactly and $\mathcal{N}(A) = \{0\}$.

The proofs of quasioptimality results, reviewed above, are based on Proposition 2. Using Proposition 3 or Proposition 1, it is possible to extend these results to the case with polluted operator.

In [20], one can find further applications of Proposition 2 to iterative and other methods. In [21, 22], for a more special problem (1), other quasioptimal versions of Tikhonov method are investigated.

VIII. OPTIMALITY ON THE SOURCE SETS

Let H and F be Hilbert spaces and $A \in \mathcal{L}(H,F)$. In this section we shall examine the optimality of regularization methods on so-called source sets

$$M_{p\rho} = \{u \in H : u = (A^*A)^{p/2}v, \ \|v\|_H \le \rho\},$$

$$M_{p\rho u_0} = \{u \in H : u - u_0 = (A^*A)^{p/2}v, \ \|v\|_H \le \rho\}, \quad u_0 \in H, \ p > 0, \ \rho > 0$$

(here $p > 0$ is a real number, not necessarily natural). The error or approximate solution we shall measure in the norm of H. In other words, in optimality definitions (see Sections 2 and 3) we put $E = H$.

Lemma 1 [7, 20]. If $(\delta/\rho)^{2/(p+1)} \in \sigma(A^*A)$ then

$$\inf_P \varphi(\delta, P, M_{p\rho u_0}, A) = \omega(\delta, M_{p\rho u_0}, A) = \rho^{1/(p+1)} \delta^{p/(p+1)}. \quad (8)$$

Note that if the range $\mathcal{R}(A) \subset F$ of A is non-closed, i.e. if problem (1) is essentially ill-posed, then spectrum $\sigma(A^*A)$ contains at least a sequence (λ_k) with $\lambda_k \to 0$, and for corresponding $\delta = \delta_k \to 0$, $(\delta_k/\rho)^{2/(p+1)} = \lambda_k$, (8) holds.

Now we define a class of regularization methods. Let $g : [0, \infty) \to \mathbb{R}$ be a differentiable function such that

$$h(\lambda) \equiv 1 - \lambda g(\lambda) > 0, \quad h'(\lambda) < 0 \quad (0 \le \lambda < \infty),$$
$$\sup_{0 \le \lambda < \infty} \lambda^p h(\lambda) < \infty \quad (0 \le p \le p_0). \quad (9)$$

Function g generates a family of functions

$$g_r(\lambda) = r g(r\lambda) \quad (0 \le \lambda < \infty) \quad (10)$$

depending on parameter $r > 0$. In case $H = F$, $A = A^* \ge 0$ we define the approximate solution of equation (1) via the formula

$$u_r = (I - A g_r(A)) u_0 + g_r(A) f_\delta \quad (11)$$

where $u_0 \in H$ is an initial approximation to the solution of equation (1), e.g. $u_0 = 0$ (then $u_r = g_r(A) f_\delta$). In the general case of non-self-adjoint problem (1) we first symmetrize it (as $A^*Au = A^*f$) and then apply a similar prescription:

$$u_r = (I - A^*A g_r(A^*A)) u_0 + g_r(A^*A) A^* f_\delta . \quad (12)$$

Theorem 1 (case $H = F$, $A = A^* \ge 0$). If, for a $p \in (0, p_0]$,

$$(p+1)\left[h^{-1}\left(\frac{1}{p+1}\right)\right]^{-2p} \lambda^{2p} h^2(\lambda) + \frac{p+1}{p}\left[h^{-1}\left(\frac{1}{p+1}\right)\right]^2 g^2(\lambda) \le 1$$
$$(0 \le \lambda < \infty) \quad (13)$$

then, for u_r defined by (11), with

$$r = h^{-1}\left(\frac{1}{p+1}\right) (\rho/\delta)^{1/(p+1)},$$

the optimal error bound on $M_{p\rho u_0}$ holds (see (8)):

$$\sup_{\substack{u \in M_{p\rho u_0}, f_\delta \in F \\ \|Au - f_\delta\| \le \delta}} \|u_r - u\|_H \le \rho^{1/(p+1)} \delta^{p/(p+1)}. \quad (14)$$

Conversely, if, for a $\lambda \in [0,\infty)$, inequality (12) is violated and if $\sigma(A) \supset [0,\varepsilon]$, $\varepsilon > 0$, then, for all sufficiently small $\delta > 0$,

$$\inf_{r>0} \sup_{\substack{u \in M_{p\rho u_0}, f_\delta \in F \\ \|Au - f_\delta\| \le \delta}} \|u_r - u\|_H > \rho^{1/(p+1)} \delta^{p/(p+1)}. \quad (15)$$

Theorem 2 (general case). If, for a $p \in (0, 2p_0]$,

$$(p+1)\left[h^{-1}\left(\frac{1}{p+1}\right)\right]^{-p} \lambda^p h^2(\lambda) + \frac{p+1}{p} h^{-1}\left(\frac{1}{p+1}\right) \lambda g^2(\lambda) \le 1 \quad (16)$$

$$(0 \le \lambda < \infty)$$

then, for u_r defined by (12), with

$$r = h^{-1}\left(\frac{1}{p+1}\right) (\rho/\delta)^{2/(p+1)},$$

the optimal error bound (14) holds. Conversely, if, for a $\lambda \in [0,\infty)$, inequality (16) is violated and if $\sigma(A^*A) \supset [0,\varepsilon]$, $\varepsilon > 0$, then, for all sufficiencly small $\delta > 0$, inequality (15) holds.

The proofs on these two theorems one can find in [20].

IX. OPTIMALITY OF LAVRENTIEV, TIKHONOV AND ITERATION METHODS ON SOURCE SETS

A. Lavrentiev and Tikhonov methods

Let $g(\lambda) = (1+\lambda)^{-1}$, then also $h(\lambda) = (1+\lambda)^{-1}$, and (9) is fulfilled with $p_0 = 1$. Denoting $\alpha = 1/r$ we can rewrite (10) in the form $g_\alpha(\lambda) = (\alpha+\lambda)^{-1}$, and, for $u_0 = 0$, approximations (11) and (12) take the forms

$$u_\alpha = (\alpha I + A)^{-1} f_\delta \quad \text{(Lavrentiev method, } A = A^* \ge 0) \quad (17)$$

and

$$u_\alpha = (\alpha I + A^*A)^{-1} A^* f_\delta \quad \text{(Tikhonov method).} \quad (18)$$

As corollaries from Theorems 1 and 2 we obtain the following optimality results (Table 1).

TABLE 1

Method	Condition for optimality on $M_{p\rho}$	Parameter choice
(17)	$0 < p \le \frac{1}{2}(\sqrt{5} - 1) \approx 0.618$	$\alpha = p^{-1}(\delta/\rho)^{1/(p+1)}$
(18)	$0 < p \le 2$	$\alpha = p^{-1}(\delta/\rho)^{2/(p+1)}$

With any parameter choice, Lavrentiev method is non-optimal on $M_{p\rho}$ for $p > \frac{1}{2}(\sqrt{5} - 1)$. (But it is known that, for $p \le 1$, with parameter choice $\alpha = d(\delta/\rho)^{1/(p+1)}$, $d = \text{const} > 0$, Lavrentiev method is quasioptimal on $M_{p\rho}$. For $p > 1$, Lavrentiev method and, for $p > 2$, Tikhonov method, with any parameter choices, are even non-quasioptimal on $M_{p\rho}$.)

B. Continuous versions of iteration methods

Let $g(\lambda) = \lambda^{-1}(1 - e^{-\lambda})$, then $h(\lambda) = e^{-\lambda}$, and (9) is fulfilled with $p_0 = \infty$. Denoting $t = r$, $u_r = u(t)$ approximations (11) and (12) can be considered as the solutions of Cauchy problems

$$u'(t) + Au(t) = f_\delta, \quad u(0) = u_0 \quad (\text{case } A = A^* \ge 0) \quad (19)$$

and

$$u'(t) + A^*Au(t) = A^*f_\delta, \quad u(0) = u_0 \quad \text{(general case).} \quad (20)$$

From Theorems 1 and 2 the following optimality results follow (see Table 2).

TABLE 2

Method	Condition for optimality on $M_{p\rho u_0}$	Parameter choice
(19)	$0 < p \leq p_1 \approx 1.043$	$t = [\ln(1+p)](\rho/\delta)^{1/(p+1)}$
(20)	$0 < p \leq p_2 \approx 7.124$	$t = [\ln(1+p)](\rho/\delta)^{2/(p+1)}$

For $p > p_1$, correspondingly $p > p_2$, methods (19) and (20) with any parameter choices, are non-optimal on $M_{p\rho u_0}$. (But, with $t = d(\rho/\delta)^{1/(p+1)}$, $d = \text{const} > 0$, correspondingly $t = d(\rho/\delta)^{2/(p+1)}$, these methods are quasioptimal on $M_{p\rho u_0}$ for any $p \in (0,\infty)$.)

C. Iteration methods

Let $\chi : [0,a] \to \mathbb{R}$ be a continuous function such that

$$0 < \chi(\lambda) < \frac{2}{\lambda} \quad (0 \leq \lambda \leq a).$$

For $A = A^* \geq 0$, $\|A\| \leq a$, denote $B = \chi(A)$; in the general case of non-selfadjoint problem (1) with $\|A\|^2 \leq a$, denote $C = \chi(A^*A)A^*$. Consider iteration methods

$$u_n = u_{n-1} - B(Au_{n-1} - f_\delta), \quad n=1,2,\ldots \quad (\text{case } A = A^* \geq 0) \qquad (21)$$

and

$$u_n = u_{n-1} - C(Au_{n-1} - f_\delta), \quad n=1,2,\ldots \quad (\text{general case}). \qquad (22)$$

Most usual forms of function $\chi(\lambda)$ are given by $\chi(\lambda) \equiv \beta$ and $\chi(\lambda) = (\beta + \lambda)^{-1}$ ($\beta = \text{const} > 0$). In case $\chi(\lambda) \equiv \beta$ iterations (21) and (22) take the form

$$u_n = u_{n-1} - \beta(Au_{n-1} - f_\delta) \quad (A = A^* \geq 0,\ 0 < \beta < \frac{2}{\|A\|}),$$

$$u_n = u_{n-1} - \beta A^*(Au_{n-1} - f_\delta) \quad (0 < \beta < \frac{2}{\|A\|^2});$$

in case $\chi(\lambda) = (\beta + \lambda)^{-1}$,

$$\beta u_n + Au_n = \beta u_{n-1} + f_\delta \quad (A = A^* \geq 0),$$

$$\beta u_n + A^*Au_n = \beta u_{n-1} + A^*f_\delta.$$

Theorems 1 and 2 cannot be applied directly to methods (21) and (22). But using some connections with continuous versions (19) and (20), it is possible to show (details see in [20]) that, for the same p as for method (19) and (20), methods (21) and (22) are asymptotically optimal on $M_{p\rho u_0}$ (see Table 3).

TABLE 3

Method	Condition for asymptotical optimality on $M_{p\rho u_0}$	Parameter choice
(21)	$0 < p \leq p_1 \approx 1.043$	$n = \text{int}\left\{\frac{\ln(1+p)}{\chi(0)}(\rho/\delta)^{1/(p+1)}\right\}$
(22)	$0 < p \leq p_2 \approx 7.124$	$n = \text{int}\left\{\frac{\ln(1+p)}{\chi(0)}(\rho/\delta)^{2/(p+1)}\right\}$

Here $\text{int}\{r\}$ denotes the integer part of r. So far as parameter n can take only natural values, optimality of iteration methods cannot be attained. (It is known that, with $n = \text{int}\{d(\rho/\delta)^{1/(p+1)}\}$, correspondingly, $n = \text{int}\{d(\rho/\delta)^{2/(p+1)}\}$, $d = \text{const} > 0$, iteration methods (21) and (22) are quasi-optimal on $M_{p\rho u_0}$ for any $p \in (0,\infty)$.)

D. Case with polluted operator

Let $\|A_\eta - A\| \leq \eta$. Sets $M_{p\rho}$ and $M_{p\rho u_0}$ depend on operator A (for operator A_η these sets take slightly another form), and Proposition 1 here cannot be applied immediately. Nevertheless, in a somewhat weakened form, the optimality results can be extended: for same values of p as in Tables 1-3, instead of the optimality (or asymptotical optimality on case of Table 3) we obtain the asymptotical

$\mathfrak{A}^+$-optimality of methods (17), (19), (21) and the asymptotical $\mathcal{L}(H,F)$-optimality of methods (18), (20), (22) on $M_{p\rho}$; in all methods we are restricted to put $u_0 = 0$. Here $\mathfrak{A}^+ \subset \mathcal{L}(H,H)$ is the set of all self-adjoint positive operators. In the parameter choices (see Tables 1-3) δ must be replaced by $\delta + \|A_\eta\|^p \rho\eta$.

X. DISCREPANCY PRINCIPLE AND QUASIOPTIMALITY ON SOURCE SETS

Let again H and F be Hilbert spaces and $A \in \mathcal{L}(H,F)$. We introduce a wider class of regularization methods than in Section VIII. Let $\{g_r(\lambda)\}_{r\in(0,\infty)}$ be a family of Borel functions $g_r : [0,a] \to \mathbb{R}$ (or $g_r : [0,a] \to \mathbb{C}$) such that

$$\sup_{0\le\lambda\le a} |g_r(\lambda)| \le \gamma r \quad (r > 0), \quad \gamma = \text{const} \quad \gamma_p = \text{const}$$
$$\sup_{0\le\lambda\le a} \lambda^p |1 - \lambda g_r(\lambda)| \le \gamma_p r^{-p} \quad (r > 0; \quad 0 \le p \le p_0). \tag{23}$$

In case $H = F$, $A = A^* \ge 0$, $\|A\| \le a$ we the define the approximate solution u_r of equation (1) via formula (11); in the general case with $A \in \mathcal{L}(H,F)$, $\|A\|^2 \le a$, we define u_r via formula (12).

If $g : [0,\infty) \to \mathbb{R}$ (or $\mathbb{C}$) is a Borel function, $\gamma = \sup_{0\le\lambda<\infty} |g(\lambda)| < \infty$, and if $\gamma_p \equiv \sup_{0\le\lambda<\infty} \lambda^p |1 - \lambda g(\lambda)| < \infty$ for $0 \le p \le p_0$, then $g_r(\lambda)$ can be defined by (10), and for this family, (23) with $a = \infty$ is fulfilled. Thus, methods (17)-(20) go also in framework (23). It turns out, that also iteration methods (21) and (22) go in framework (23) with $p_0 = \infty$; for further examples see [20].

Now we shortly review some results [20] about the choice of regularization parameter r by means of the discrepancy

principle. First note that for both approximations, (11) and (12), $\lim_{r\to\infty} \|Au_r - f_\delta\| = \inf_{u\in H} \|Au - f_\delta\| \le \delta$ (the last inequality under condition $f \in \mathcal{R}(A)$). In case $H = F$, $A = A^* \ge 0$ we choose any $r = r(\delta)$ such that, for approximate solution (11),

$$b_1\delta \le \|Au_r - f_\delta\| \le b_2\delta \quad \text{with} \quad 1 < b_1 \le b_2 \tag{24}$$

(value $b_1 = 1$ leads to the divergence of method (11)!). If (23) is fulfilled with a $p_0 > 1$ then thus we obtain a quasioptimal method on $M_{p\rho u_0}$ for $0 < p \le p_0 - 1$ (and not for $0 < p \le p_0$ as by appropriate a priori choice of parameter!). In other words, for $0 < p \le p_0 - 1$,

$$\sup_{u\in M_{p\rho u_0},\, f_\delta\in F,\, \|Au - f_\delta\|\le\delta} \|u_r - u\|_H \le c_p\, \rho^{1/(p+1)} \delta^{p/(p+1)} \tag{25}$$

where c_p is the quasioptimality constant (its value one can find in [20]).

In case $A \in \mathcal{L}(H,F)$ and approximate solution (12), similar results hold, but now condition $p_0 > 1/2$ is sufficient, and quasioptimality on $M_{p\rho u_0}$ (error estimate (25)) holds for $0 < p \le 2p_0 - 1$. Value $b_1 = 1$ in (24), for approximate solution (12), is also allowable, if instead of (23) the following strengthened condition holds:

$$g_r(\lambda) \ge 0, \; 0 \le 1 - \lambda g_r(\lambda) \le g_r(\lambda)/\varkappa_r \; (0\le\lambda\le a), \; \varkappa_r = \sup_{0\le\lambda\le a} g_r(\lambda),$$
$$\beta r \le \varkappa_r \le \gamma r \quad (r > 0), \quad \beta = \text{const} > 0, \quad \gamma = \text{const} \tag{26}$$

(note that (23) with $p_0 = 1$ is a consequence from (26)). Methods (18) and (20) go in framework (26) also; iteration method (22) goes if

$$\chi(\lambda) \geq 0, \quad 0 \leq 1 - \lambda\chi(\lambda) \leq \chi(\lambda)/\varkappa \quad (0 \leq \lambda \leq a), \qquad (27)$$
$$\varkappa = \sup_{0 \leq \lambda \leq a} \chi(\lambda).$$

For $0 < p \leq 1$, under condition (26) and discrepancy level $\|Au_r - f_\delta\| = \delta$, quasioptimal error estimate (25) with $c_p = 2^p$ holds.

For iteration methods, the following version of discrepancy principle is preferable: stop iterations (21) or (22) on the first $n = n(\delta)$ for which

$$\|Au_n - f_\delta\| \leq b\delta$$

($b > 1$ in case of iterations (21) or (22); $b \geq 1$ in case of iterations (22) under condition (27)). Similar quasioptimality results us above hold.

In case of polluted operator, there are several versions of the discrepancy principle which yield $\mathfrak{A}^+$-quasioptimal or $\mathcal{L}(H,F)$-quasioptimal methods on $M_{p\rho u_0}$. For details, see [16, 17, 20].

Let us emphasize the following essential circumstance. From the algorithmical point of view, the versions of the discrepancy principle considered above need no information about the exact solution of equation (1). When we are choosing r, we need not know a class $M_{p\rho u_0}$ which, maybe, contains the exact solution. But if it does contain, then the discrepancy choice of r is quasioptimal one for any $M_{p\rho u_0}$ $(0 < p \leq p_0 - 1$ or $0 < p \leq 2p_0 - 1)$.

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