On the Choice of the Stopping Index in Iteration Methods for Solving Problems with Noisy Data

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Abstract

We consider linear ill-posed and well-posed problems with noisy right hand side and given noise level. To solve equations iteration methods are considered. For choice of the appropriate stopping index two rules are discussed, which guarantee for the approximate solution quasioptimal error estimate.

Keywords

Ill-posed problem, iterative methods, stopping rule, quasi-optimality, error estimates, noisy data, Landweber method, implicit iteration.

I. Introduction

We consider linear problem

$$Au = f \tag{1}$$

where $A \in \mathcal{L}(H, F)$ is bounded operator and H, F are real Hilbert spaces. We do not assume that range $\mathcal{R}(A)$ is closed and the kernel $\mathcal{N}(A)$ is trivial, therefore the problem (1) may be ill-posed. We assume that instead of $f \in \mathcal{R}(A)$ only noisy element $f_\delta \in F$ with $\|f_\delta - f\| \le \delta$ and known noise level δ is available.

To solve problem (1) we consider iterative methods in the form

$$u_n = u_{n-1} - A^* z_{n-1} \quad (n = 1, 2, \ldots) \tag{2}$$

with $z_n \in F$ and initial guess $u_0 \in H$. The elements z_n characterize the special iteration method. Our main attention is given to choice $z_n = g(AA^*) r_n$, where $r_n = Au_n - f_\delta$ and $g : [0, a] \to I\!R$ with $a = \|A\|^2$ is a continuous function satisfying

$$0 \le g(\lambda) \le \lambda^{-1} \quad \text{for} \quad 0 \le \lambda \le a. \tag{3}$$

This choice of z_n gives methods

$$u_n = u_{n-1} - g(A^*A)A^*(Au_{n-1} - f_\delta) \quad (n = 1, 2, \ldots). \tag{4}$$

Special cases of iterative methods (4) are the explicit iteration scheme (called also as Landweber's method)

$$u_n = u_{n-1} - \mu A^*(Au_{n-1} - f_\delta), \quad n = 1, 2, \ldots \tag{5}$$

with constant function $g(\lambda) = \mu = \text{const} \in (0, \|A\|^{-2})$ and the implicit iteration scheme

$$\alpha u_n + A^*Au_n = \alpha u_{n-1} + A^*f_\delta, \quad n = 0, 1, \ldots \tag{6}$$

with function $g(\lambda) = (\alpha + \lambda)^{-1}$, where $\alpha = \text{const} > 0$.

Note that often for solution of ill-posed problems regularization methods are considered in form

$$u_r = u_0 + g_r(A^*A)A^*(f_\delta - Au_0) \quad (r \in I\!R, r > 0) \tag{7}$$

(see [7], [8]), where (piecewise) continuous functions $g_r(\lambda) : [0, a] \to I\!R$ satisfy condition

$$\sup_{0<\lambda<a} |\sqrt{\lambda} g_r(\lambda)| \le \overline{\gamma}\sqrt{r} \quad (r > 0). \tag{8}$$

Iteration methods (4) may also be considered in form (7) with $r = n$ and with

$$g_r(\lambda) = \lambda^{-1}(1 - (1 - \lambda g(\lambda))^r). \tag{9}$$

The important question in solving problem (1) by methods (2), (4) is appropriate choice of the stopping index n. For method (4) in case of exact data ($\delta = 0$) it is known the monotone convergence $\|u_n - u_*\| \to 0$ $(n \to \infty)$, where u_* is nearest to u_0 solution of (1) (see [7], [8]). In case of noisy data ($\delta > 0$) the monotone decrease of errors $\|u_n - u_*\|$ with respect of n can be proved only for small n, typically the errors for great n are essentially larger as for optimal n. In this paper we consider two rules for choice of stopping index in method (4) and give for corresponding approximate solutions the quasioptimal error estimate.

Note that by solving well-posed problems rules for stopping the iterations (cf [1]) often ignore noise in data but our error estimates show that recommended here stopping rules can be useful in well-posed problems as well.

II. Stopping rules and monotonicity of error

We consider following two stopping rules. We use notation $r_n = Au_n - f_\delta$.

Discrepancy principle [7], [8]. Let $b > 1$ be a constant. We choose $n = n_D$ as the first index $n = 1, 2, \ldots$ for which

$$\|Au_n - f_\delta\| < b\delta. \tag{10}$$

Note that in iterative methods (4) by condition (3) the discrepancy $\|Au_n - f_\delta\|$ is the monotonically decreasing function of index n and $\lim_{n\to\infty} \|Au_n - f_\delta\| < \delta$ (see [7], [8]), hence the existence of n_D is guaranteed.

Monotone error rule (ME-rule) [2] (see also [4]). Choose n_{ME} in (2) as the first index $n = 0, 1, \ldots$ for which

$$d_{ME}(n) = \frac{(r_n + r_{n+1}, z_n)}{2\|z_n\|} < \delta\,. \tag{11}$$

The form of function $d_{ME}(n)$ in special methods depends on corresponding element z_n in (2). In method (4) $z_n = g(AA^*)r_n$ and hence

$$d_{ME}(n) = \frac{(r_n + r_{n+1}, g(AA^*)r_n)}{2\|g(AA^*)r_n\|}\,. \tag{12}$$

In next theorem we study, to which iteration index the monotonical decrease of the error of approximate solution is guaranteed.

Theorem 1: The ME-rule gives in iterative methods (2) the stopping index n_{ME} with property

$$\|u_n - u_*\| < \|u_{n-1} - u_*\| \quad (n = 1, 2, \ldots, n_{ME}). \tag{13}$$

In iterative method (4) with $g(\lambda)$ satisfying

$$\frac{1}{c+\lambda} < g(\lambda) < \frac{1}{\min(\lambda, c)} \quad \text{for some } c > 0 \tag{14}$$

the discrepancy principle gives the stopping index n_D with property

$$\|u_n - u_*\| < \|u_{n-1} - u_*\| \quad (n = 1, 2, \ldots, n_D - 1). \tag{15}$$

Proof: From (2) and from the inequality $\|f_\delta - f\| < \delta$ we have

$$\|u_n - u_*\|^2 - \|u_{n-1} - u_*\|^2 = (2u_* - u_{n-1} - u_n, A^* z_{n-1})$$
$$= (2(f - f_\delta) - (r_{n-1} + r_n), z_{n-1}) < 2\|z_{n-1}\|\left\{\delta - \tfrac{(r_{n-1}+r_n, z_{n-1})}{2\|z_{n-1}\|}\right\}$$

According to the ME-rule (see (11)) the expansion in brackets $\delta - d_{ME}(n-1) < 0$ for $n = 1, 2, \dots, n_{ME}$, hence (13) holds. The property (15) for discrepancy principle follows from (13) and from fact that in iterative methods (4) by given assumptions ME-rule and discrepancy principle are similar in the following sense:

$$\|r_{n+1}\| < d_{ME}(n) < \|r_n\| ; \tag{16}$$

if $b = 1$, then

$$n_D - 1 < n_{ME} < n_D . \tag{17}$$

It is sufficient to show property (16), while in case $b = 1$ property (17) follows from (16). From (3), (14) we have right inequality (16)

$$d_{ME}(n) < \|r_n + r_{n+1}\|\|/2 = \|(2I - AA^*g(AA^*))r_n\|/2 < \|r_n\| .$$

The right and left inequalities in (14) give the inequalities

$$c\|g(AA^*)r_n\|^2 < (r_n, g(AA^*)r_n) , \quad c^{-1}\|r_{n+1}\|^2 < (r_{n+1}, g(AA^*)r_n)$$

respectively. Using them and inequality of type $2ab < a^2 + b^2$ we get

$$2\|g(AA^*)r_n\|\|r_{n+1}\| < c\|g(AA^*)r_n\|^* + \frac{1}{c}\|r_{n+1}\|^2 < (r_n + r_{n+1}, g(AA^*)r_n).$$

This gives left inequality in (16). ■

Note that assumption (14) is fulfilled in method (5) with $c = \mu^{-1}$ and in method (6) with $c = \alpha$.

III. Quasioptimality of the discrepancy principle

In the following we show that discrepancy principle leads in iterative methods (4) to optimal convergence rates for every concrete problem $Au = f$. For this we compare the error of approximate solution $\|u_{n_D} - u_*\|$ with quantity

$$M(\delta) = \sup_{\tilde{f}\in F, \|\tilde{f}-f\|<\delta} \inf_{n\geq 0} \|\tilde{u}_n - u_*\| ,$$

where $\tilde{u}_n$ is the approximate solution given by formula (4) with $\tilde{f}$ instead of f_δ. The quantity $M(\delta)$ is the smallest error of solutions in iterative method (4) among all right-hand sides with noise level δ.

Theorem 2: Let the function $g(\lambda)$ satisfy (13) and corresponding function $g_r(\lambda)$ in (9) satisfy (8). Then for method (4) discrepancy principle gives stopping index n_D such that

$$\|u_{n_D} - u_*\| < (b+2)M(\delta) + \overline{\gamma}\delta\,. \tag{18}$$

To prove Theorem 2 we need some preparations. As intermediate step we need some results for continuous regularization method (7). It is easy to verify that for functions (9), for which $g(\lambda)$ satisfies (3) the following properties hold:

1° $g_r(\lambda) > 0$, $1 - \lambda g_r(\lambda) > 0$, $(0 < \lambda < a,\ r > 0)$;

2° the function $1 - \lambda g_r(\lambda)$ is monotone decreasing with respect r;

3° the function $g_r(\lambda)$ is monotone increasing with respect r;

4° $\lim\limits_{r\to\infty} \lambda g_r(\lambda) = 1$;

5° $g_{r_2}(\lambda) - g_{r_1}(\lambda) < g_{r_2-r_1}(\lambda)$ $(0 < \lambda < \alpha,\ r_2 > r_1 > 0)$.

The property 5° follows from inequalities

$$g_{r_2}(\lambda) - g_{r_1}(\lambda) = (1 - \lambda g(\lambda))^{r_1} \cdot \tfrac{1}{\lambda}[1 - (1 - \lambda g(\lambda))^{r_2-r_1}$$
$$\le \tfrac{1}{\lambda}[1 - (1 - \lambda g(\lambda))^{r_2-r_1} \quad g_{r_2-r_1}(\lambda)\,.$$

In the following two lemmas we give for the quantity $M(\delta)$ two lower bounds. We will use notations $K_n = I - A^*Ag_n(A^*A)$, $v = u_0 - u_*$, $G_n = g_n(A^*A)A^*$.

Lemma 1: It holds

$$M(\delta) > \inf_{n>0}\{\|K_n(u_0 - u_*)\|^2 + \|g_n(A^*A)A^*(f_\delta - f)\|^2\}^{1/2}.$$

Proof: Let $Q(\lambda)$ and $P(\lambda)$ be the spectral families of projectors of operators AA^* and A^*A respectively. Then we can construct the element $f_\delta^0 \in F$ such that for $\lambda \in 0, a$

$$(Q(\lambda)(f_\delta - f), f_\delta - f) = (Q(\lambda)(f_\delta^0 - f), f_\delta^0 - f), \quad (P(\lambda)(u_0 - u_*), A^*(f_\delta^0 - f)) > 0.$$

Denoting $u_n^0 = u_0 + g_n(A^*A)A^*(f_\delta^0 - Au_0)$ we have

$$M^2(\delta) > \inf_{n>0} \|u_n^0 - u_*\|^2 = \inf_{n>0} \|K_n v + G_n(f_\delta^0 - f)\|^2$$
$$= \inf_{n>0} \{\|K_n v\|^2 + \|G_n(f_\delta^0 - f)\|^2 + 2\mathcal{R}e(K_n v, G_n(f_\delta^0 - f))\}$$
$$> \inf_{n>0} \{\|K_n v\|^2 + \|G_n(f_\delta^0 - f)\|^2\} = \inf_{n>0} \{\|K_n v\|^2 + \|G_n(f_\delta - f)\|^2\}\,,$$

which proves the lemma 1. ■

Lemma 2: Let the parameter $r(c) \in I\!R$ be such that

$$\|AK_{r(c)}(u_0 - u_*)\| = c\,\delta, \quad c > 1. \tag{19}$$

Then it holds

$$M(\delta) \geq \|K_{r(c)}(u_0 - u_*)\|/c\,.$$

Proof: We construct the element $f^c = f + c^{-1}AK_{r(c)}(u_0 - u_*),\ f^c \in F$. By assumption (19) $\|f^c - f\| = \delta$ and therefore

$$\inf_{r>0} \|u_r^c - u_*\| \leq M(\delta)\,, \tag{20}$$

where $u_r^c = u_0 + g_r(A^*A)A^*(f^c - Au_0)$. In the following we show that

$$\inf_{r>0} \|u_r^c - u_*\| = c^{-1}\|K_{r(c)}(u_0 - u_*)\|\,, \tag{21}$$

which with (20) proves the lemma.

We have

$$u_r^c - u_* = u_0 + g_r(A^*A)A^*(f^c - Au_0) - u_* = K_r v + G_r(f^c - f)$$

$$= K_r v + c^{-1}G_r AK_{r(c)}(u_0 - u_*) = [K_r + c^{-1}G_r AK_{r(c)}](u_0 - u_*)\,.$$

Using property 1° and inequality $c > 1$ we can estimate

$$\|u_r^c - u_*\| \geq c^{-1}\|(K_r + A^*Ag_r(A^*A)K_{r(c)})(u_0 - u_*)\|$$

$$= c^{-1}\|(I - (A^*A)^2 g_r(A^*A)g_{r(c)}(A^*A))(u_0 - u_*)\|\,.$$

The function $1 - \lambda^2 g_r(\lambda)g_{r(c)}(\lambda)$ is monotone decreasing with respect r (see properties 1° and 3°) from which with property 4° follows that

$$\inf_{r>0} \|u_r^c - u_*\| = \lim_{r\to\infty} c^{-1}\|(I - (A^*A)^2 g_r(A^*A)g_{r_c}(A^*A))(u_0 - u_*)\|$$

$$= c^{-1}\|(I - A^*Ag_{r(c)}(A^*A))(u_0 - u_*)\| = c^{-1}\|K_{r(c)}(u_0 - u_*)\|\,.$$

This proves the lemma 2. ■

Proof: (Proof of Theorem 2.) At first we give some auxiliary results. We have

$$u_n - u_* = K_n(u_0 - u_*) + g_n(A^*A)A^*(f_\delta - f), \tag{22}$$

$$Au_n - f_\delta = AK_n(u_0 - u_*) + \bar{K}_n(f_\delta - f)\,,$$

where $K_n = I - AA^* g_r(AA^*)$. Due to inequality $\|Au_{n_D} - f_\delta\| < b\delta$ we obtain

$$\|AK_{n_D}(u_0 - u_*)\| < \|Au_{n_D} - f_\delta\| + \|K_{n_D}(f_\delta - f)\| < (b+1)\delta\,. \tag{23}$$

Let $n_* \in I\!N$ be a parameter for which the function

$$\Psi(n) = \{\|K_n(u_0 - u_*)\|^2 + \|g_n(A^*A)A^*(f_\delta - f)\|^2\}^{1/2}$$

has a global minimum. From lemma 1 then follows that

$$M(\delta) > \Psi(n_*)\,. \tag{24}$$

To prove the estimation (18) we consider separately two cases: 1) $n_* < n_D - 1$, 2) $n_* > n_D$.

1. Case $n_* < n_D - 1$. Using Theorem 1 we get

$$\|u_{n_D} - u_*\| < \|u_{n_D-1} - u_*\| + \|u_{n_D} - u_{n_D-1}\| < \|u_{n_*} - u_*\| + \|u_{n_D} - u_{n_D-1}\|\,. \tag{25}$$

From (22), (24) it follows that

$$\|u_{n_*} - u_*\| < \|K_{n_*}(u_0 - u_*)\| + \|g_{n_*}(A^*A)A^*(f_\delta - f)\| < \sqrt{2}M(\delta). \tag{26}$$

Using properties 5° and 2° and (22), (24), (8) we get

$$\begin{aligned}
\|u_{n_D} - u_{n_D-1}\| &= \|(u_{n_D} - u_*) - (u_{n_D-1} - u_*)\| \\
< \|(K_{n_D-1} - K_{n_D})(u_0 - u_*)\| &+ \|(g_{n_D}(A^*A) - g_{n_D-1}(A^*A))A^*(f_\delta - f)\| \qquad (27)\\
< \|K_{n_D-1}(u_0 - u_*)\| &+ \delta \cdot \sup_{0<\lambda<a} \sqrt{\lambda}g_1(\lambda) < M(\delta) + \overline{\gamma}\delta\,.
\end{aligned}$$

Now from (25), (26), (27) follows the estimation

$$\|u_{n_D} - u_*\| < (\sqrt{2}+1)M(\delta) + \overline{\gamma}\delta\,. \tag{28}$$

2. Case $n_* > n_D$. From properties 1° and 2° follows that the function $d(r) = \|AK_r(u_0 - u_*)\|$ is monotone decreasing and therefore $n_D > r(b+1)$, where $r(b+1)$ is the parameter for which $\|AK_{r(b+1)}(u_0 - u_*)\| = (b+1)\delta$. Using lemma 2 we get

$$\begin{aligned}
\|u_{n_D} - u_*\| &< \|K_{n_D}(u_0 - u_*)\| + \|g_{n_D}(A^*A)A^*(f_\delta - f)\| \\
&< \|K_{r(b+1)}(u_0 - u_*)\| + \|g_{n_*}(A^*A)A^*(f_\delta - f)\| \\
&< (b+1)M(\delta) + \Psi(n_*) < (b+2)M(\delta)\,.
\end{aligned}$$

which with (28) proves the theorem. ■

Remark 1: If in iterative methods (5), (6) the stopping index is choosed by discrepancy principle, some error estimates with use of quantity $M(\delta)$ were earlier given in [6]. However, in [6] the coefficient $c(b)$ before $M(\delta)$ was dependent from the spectrum of the operator A^*A and also $c(b) \to \infty$ for $b \to 1$.

Remark 2: One can prove that for ME-rule in iterative methods (5), (6) the following analogue of estimate (18) holds:

$$\|u_{n_{ME}} - u_*\| \le 3M(\delta) + c\,\delta\,.$$

Here $c = 3$ for method (5), $c = 2$ for method (6).

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References

1 O. Axelsson. *Iterative solution methods.* Cambridge University Press, Cambridge, 1994.

2 U. Hämarik. *Monotonicity of error and choice of the stopping index in iterative regularization methods.* In Differential and Integral Equations: Theory and Numerical Analysis, A. Pedas ed., Estonian Math. Society, Tartu, 1999, pp. 15–30.

3 U. Hämarik and T. Raus. *On the a posteriori parameter choice in regularization methods.* Proc. Estonian Acad. Sci. Phys. Math., 48 (1999), pp. 133–145.

4 U. Hämarik and U. Tautenhahn. *The monotone error rule for parameter choice in regularization methods.* In ENUMATH 99 – Proc. of the 3rd European Conf. on Numer. Math. and Advanced Appl., P. Neittaanmäki, T. Tiihonen and P. Tarvainen eds., World Scientific, Singapore, 2000, pp. 518–525.

5 T. Raus. *Residue principle for ill-posed problems.* Acta et Comment. Univ. Tartuensis, 672 (1984), pp. 16–26.

6 T. Raus. *On the discrepancy principle for solution of ill-posed problems with non-selfadjoint operators.* Acta et Comment. Univ. Tartuensis, 715 (1985), pp. 12–20.

7 G.M. Vainikko. *The discrepancy principle for a class of regularization methods.* USSR Comp. Math. Math. Phys., 22 (1982), pp. 1–19.

8 G.M. Vainikko and A.Y. Veretennikov. *Iteration Procedures in Ill-Posed Problems.* Nauka, Moscow, 1986 (in Russian).