On self-regularization of ill-posed problems in Banach spaces by projection methods

Uno Hämarik and Urve Kangro

Abstract We consider ill-posed linear operator equations with operators acting between Banach spaces. For the stable solution of ill-posed problems regularization is necessary, and for using computers discretization is necessary. In some cases discretization may also be used as regularization method with discretization parameter as regularization parameter, additional regularization is not needed. Regularization by discretization is called self-regularization. We consider self-regularization by projection methods, giving necessary and sufficient conditions for self-regularization by a priori choice of the dimension of subspaces as the regularization parameter. Convergence conditions are also given for the choice of the dimension by the discrepancy principle, without the requirement that the projection operators are uniformly bounded.

1 Introduction

Consider an ill-posed linear operator equation

$$Au = f, \quad f \in \mathscr{R}(A) \tag{1}$$

where $A \in L(E,F)$ ia a linear injective mapping between nontrivial Banach spaces E and F. In practice only noisy data f^δ will be given. We assume here that the noise level δ satisfying

$$\|f^\delta - f\| \leq \delta \tag{2}$$

is known. For the stable solution of problem (1) it will be regularized to guarantee the convergence of regularized solutions to an exact solution u_* of (1) as δ goes to zero (see [9, 34]). Often ill-posed problems are formulated in infinite-dimensional space, but for using computers the problem will be discretized, leading to some

University of Tartu, e-mail: uno.hamarik@ut.ee, urve.kangro@ut.ee

finite-dimensional (n-dimensional) problem. Typically discretization and regularization are used as separate procedures (see [14] for error estimates in regularized projection methods). However, if the data are exact, the successful discretization can lead to well posed problem with unique solution, which may converge to the solution of the original infinite-dimensional problem, if the dimensions of the discretized problems tend to infinity (see [20] for convergence conditions of projection methods). In this situation the self-regularization is possible: if data are noisy with known noise level δ, then by proper choice of $n = n(\delta)$ the solutions of discretized equations with noisy data converge to the solution of the original problem with exact data.

Self-regularization is probably the oldest regularization method. It is folklore of numerics that in numerical differentiation of a given noisy function by finite difference scheme, the discretization step h as the regularization parameter should be chosen in dependence of the noise level (see e.g. [9, 26]). From 1972 it is known (see [2]) that the quadrature formula method is a self-regularization method for the solution of the Volterra integral equation of the first kind; the rules for choice of the discretization step $h = h(\delta)$ as the regularization parameter in dependence of noise in the kernel and in the right-hand were given in [2] (see also [1]).

In this paper we consider projection methods. Let $E_n \subset E$, $Z_n \subset F^*$, $n \in \mathbb{N}$, be finite-dimensional nontrivial subspaces which have the role of approximating the spaces E and F^*, respectively. The general projection method defines a finite-dimensional approximation u_n to u_* by

$$u_n \in E_n \text{ and } \forall z_n \in Z_n : \langle z_n, Au_n \rangle_{F^*,F} = \langle z_n, f^\delta \rangle_{F^*,F}. \tag{3}$$

We also consider the least squares method (the “least residual” method would be a more natural name)

$$u_n \in \operatorname{argmin}\{ \|A\tilde{u}_n - f^\delta\|_F : \tilde{u}_n \in E_n \} \tag{4}$$

and the least error method

$$u_n \in \operatorname{argmin}\{ \|\tilde{u}\|_E : \forall z_n \in Z_n : \langle z_n, A\tilde{u} \rangle_{F^*,F} = \langle z_n, f^\delta \rangle_{F^*,F} \}. \tag{5}$$

The name “least error” method is justified by the fact that the obtained approximation u_n satisfies in case of exact data the inequalities

$$\|u_* - u_n\| \le \|u_* - v_n\|, \quad D(u_*, u_n) \le D(u_*, v_n) \quad \forall v_n \in E_n$$

in Hilbert and Banach spaces respectively (see [16, 32]), where $D(u_*, u_n)$ is the Bregman distance and $E_n \subset E$ is a certain subspace. It means that u_n is respectively the orthogonal projection or Bregman projection of u_* onto E_n. This method is called dual least-squares method in [3, 9, 21, 23, 26] and moment method in [21]. If E, F are Hilbert spaces, the least squares and least error methods are characterized by the equalities $Z_n = AE_n$ and $E_n = A^*Z_n$ respectively. If $E = F$ is a Hilbert space and $A = A^* \ge 0$, Galerkin method $E_n = F_n$ also can be used. Approximate solutions of the least squares and least error methods are found from a system of equations which is linear in Hilbert spaces and unfortunately nonlinear in Banach spaces.

In the collocation method, $Z_n = \text{span}\{\delta(t-t_i), i = 1, \ldots, n\}$ consists of linear combinations of Dirac's δ-functions $\delta(t-t_i)$ with support at collocation points $t_i, i = 1, \ldots, n$. Then (3) are the collocation conditions

$$u_n \in E_n, \qquad Au_n(t_i) = f^\delta(t_i), i = 1, \ldots, n \tag{6}$$

for finding u_n from arbitrary fixed subspace E_n.

Use of $Z_n = \text{span}\{\delta(t-t_i), i = 1, \ldots, n\}$ in the least error method (5) gives the least error collocation method, called also least-squares collocation [8, 9, 25] or moment collocation [21]. This method uses also collocation conditions (6), but the approximate set E_n is not arbitrary, it results from the condition that u_n is a minimum-norm solution of equation (6). If E is Hilbert space, then E_n is a subspace of E, but if E is a Banach space, then E_n is not necessarily a linear space.

Self-regularization by projection method was studied in [26], where the error estimates were given in Banach space formulation, convergence conditions were given for the collocation method, in Hilbert space formulation also for least squares and least error methods. The error estimates there (in Sobolev space formulation for least squares and Galerkin method also in [27]) allow to formulate a priori rules for the choice of dimension $n = n(\delta)$. For operator equations with noisy operator and noisy right-hand side the least squares, least error and Galerkin method were studied with a priori parameter choice in [12] and with a posteriori choice via discrepancy principle in [13]. Necessary and sufficient conditions for regularization by general projection methods in Hilbert spaces were given in [32], applications to mentioned methods and to class of integral equations of the first kind with Green type kernels were given. Convergence of the least error collocation method in case of exact data was proved for the space $E = L_2$ in [25, 8, 21], for Sobolev space $E = W_2^m$ in [33], for a priori choice $n = n(\delta)$ in [8], for a posteriori choice $n = n(\delta)$ by the monotone error rule in [15]. In the least error method in Hilbert spaces, a posteriori choice by the monotone error rule was studied in [15, 10], by the balancing principle in [3] (these both rules need weaker assumptions than the discrepancy principle). In Banach spaces the discrepancy principle was studied in [21] for a method close to the least squares method, in [16] for the general projection method and for the least squares method. Error estimates in Sobolev and Hölder-Zygmund norms of various discretization methods in certain boundary integral equations with a priori choice of $n = n(\delta)$ were given in [4]. Convergence of collocation method in case of exact data was analysed in [6, 7, 5], convergence by choice of $n = n(\delta)$ by discrepancy principle was proved in [16]. See also other works [11, 17, 19, 24, 22] about self-regularization.

In this paper we consider in Section 2 the general projection method. The necessary and sufficient conditions for self-regularization by a priori choice $n = n(\delta)$ are given. Our approach is similar to [21], instead of a projector we use operator Q_n defined by (7). In previous treatments of the a posteriori choice of $n = n(\delta)$ by the discrepancy principle it was required that the projection operators are uniformly bounded. We modify the discrepancy principle so that this requirement is not needed. In Section 3 we consider the least squares method, in Section 4 the collocation method, where also numerical examples are given.

2 The general projection method

Let Q_n be the linear operator defined by

$$Q_n : F \to Z_n^* \quad \forall g \in F\,, z_n \in Z_n : \, \langle Q_n g, z_n \rangle_{Z_n^*, Z_n} = \langle z_n, g \rangle_{F^*, F} \tag{7}$$

which allows us to write (3) as

$$u_n \in E_n \quad \text{and} \quad Q_n A u_n = Q_n f^\delta. \tag{8}$$

The norm of Q_n equals one since

$$\|Q_n\| = \sup_{g\in F,\, \|g\|_F=1} \|Q_n g\|_{Z_n^*} = \sup_{g\in F,\, \|g\|_F=1, z_n\in Z_n,\, \|z_n\|_{F^*}=1} \langle Q_n g, z_n \rangle_{Z_n^*, Z_n} =$$

$$= \sup_{g\in F,\, \|g\|_F=1, z_n\in Z_n,\, \|z_n\|_{F^*}=1} \langle z_n, g \rangle_{F^*, F} = 1.$$

In the following lemma from [16] we give conditions under which the operator $A_n := Q_n A|_{E_n} : E_n \to Z_n^*$ has an inverse, the quantities

$$\kappa_n := \sup_{v_n\in E_n} \frac{\|v_n\|_E}{\|Av_n\|_F}, \quad \breve{\kappa}_n := \|A_n^{-1} Q_n\|, \quad \tilde{\kappa}_n := \|A_n^{-1}\| = \sup_{v_n\in E_n} \frac{\|v_n\|_E}{\|Q_n A v_n\|_F}, \tag{9}$$

$$\tau_n := \sup_{v_n\in E_n, v_n\neq 0} \frac{\|Av_n\|_F}{\|Q_n A v_n\|_{Z_n^*}}. \tag{10}$$

are finite and u_n from (3) is well-defined.

Lemma 1. *Let*

$$\mathrm{dim(E_n)} = \mathrm{dim(Z_n)} \tag{11}$$

and

$$\mathscr{N}(Q_n A) \cap E_n = \{0\} \tag{12}$$

hold. Then the operator A_n has an inverse and (3) is uniquely solvable for any $f^\delta \in F$. We have the inequalities

$$\kappa_n \le \breve{\kappa}_n \le \tilde{\kappa}_n \le \tau_n \kappa_n. \tag{13}$$

If

$$\exists \tau < \infty : \quad \tau_n \le \tau \quad \forall n \in \mathbb{N} \tag{14}$$

then also

$$\tilde{\kappa}_n \leq \tau \kappa_n,$$

i.e. the quantities κ_n, $\breve{\kappa}_n$ *and* $\tilde{\kappa}_n$ *are all equivalent as* $n \to \infty$.

Remark 1. If $\mathscr{R}(A) \neq \overline{\mathscr{R}(A)}$ and the subspaces E_n satisfy the condition

$$\inf_{v_n \in E_n} \|v_n - v\| \to 0 \quad \forall v \in E \text{ as } n \to \infty, \tag{15}$$

then A^{-1} is unbounded and $\kappa_n \to \infty$ as $n \to \infty$.

2.1 Convergence with a priori choice of n

Theorem 1. *Let the operator A be injective. Let for $n \geq n_0$ the assumptions (11), (12) be satisfied. Then for $n \geq n_0$ the projection method (3) defines the unique approximation u_n, and the following error estimate holds:*

$$\begin{aligned} \|u_n - u_*\|_E &\leq \min_{v_n \in E_n} [\, \|u_* - v_n\|_E + \|A_n^{-1} Q_n A(u_* - v_n)\|_E] + \breve{\kappa}_n \delta \\ &\leq (1 + \|A_n^{-1} Q_n A\|)\, dist(u_*, E_n) + \breve{\kappa}_n \delta. \end{aligned} \tag{16}$$

In case of exact data ($\delta = 0$) the convergence

$$\|u_n - u_*\|_E \to 0 \text{ as } n \to \infty \tag{17}$$

holds if and only if there exists a sequence of approximations $(\hat{u}_n)_{n \in \mathbb{N}}$, $\hat{u}_n \in E_n$, satisfying the convergence conditions

$$\|u_* - \hat{u}_n\|_E \to 0 \text{ as } n \to \infty \tag{18}$$

and

$$\|A_n^{-1} Q_n A(u_* - \hat{u}_n)\| \to 0 \text{ as } n \to \infty. \tag{19}$$

If these conditions hold and the data are noisy, then choosing $n = n(\delta)$ according to a priori rule

$$n(\delta) \to \infty \text{ and } \breve{\kappa}_{n(\delta)} \delta \to 0 \text{ as } \delta \to 0 \tag{20}$$

we have convergence

$$\|u_{n(\delta)} - u_*\|_E \to 0 \text{ as } \delta \to 0. \tag{21}$$

Proof. For any $v_n \in E_n$ we have, due to linearity of A,

$$\|u_n - u_*\|_E \leq \|u_* - v_n\|_E + \|u_n - v_n\|_E = \|u_* - v_n\|_E + \|A_n^{-1} Q_n (f^\delta - A v_n)\|_E =$$

$$\begin{aligned} &= \|u_* - v_n\|_E + \|A_n^{-1} Q_n [A(u_* - v_n) + f^\delta - f]\|_E \leq \\ &\leq \|u_* - v_n\|_E + \|A_n^{-1} Q_n A(u_* - v_n)\|_E + \breve{\kappa}_n \delta, \end{aligned}$$

hence the convergence estimate (16) holds.

If (18), (19) hold, then estimate (16) with $v_n = \hat{u}_n$ and our assumptions on the choice of $n(\delta)$ give convergence in both cases $\delta = 0$ and $\delta > 0$.
To show the necessity of (18), (19), note that if $\delta = 0$ and the convergence (17) holds, then $\hat{u}_n = u_n$ satisfies (18) and (19) (then $A_n^{-1}Q_nA(u_* - \hat{u}_n) = u_n - \hat{u}_n = 0$). □

According to the previous theorem in case $\delta = 0$ convergence (17) may hold due to sufficient smoothness of the solution. From this theorem we get in the following theorem conditions for convergence for every $f \in \mathscr{R}(A)$ (i.e. for every $u_* \in E$ without additional smoothness requirements).

Theorem 2. *Let the operator A be injective. Let for $n > n_0$ the assumptions (11), (12) be satisfied. Then in case of exact data ($\delta = 0$) the convergence (17) holds for every $f \in \mathscr{R}(A)$ if and only if the subspaces E_n satisfy condition (15) and the projectors $A_n^{-1}Q_nA : E \to E_n$ are uniformly bounded, i.e.,*

$$\|A_n^{-1}Q_nA\| \le M \tag{22}$$

for all $n \ge n_0$ and some constant M.
The last two conditions are necessary and sufficient for existence of relations $n = n(\delta)$ for convergence (21) for every $f \in \mathscr{R}(A)$ given approximately as arbitrary f^δ with $\|f^\delta - f\| \le \delta$.

Proof. At first we show that conditions (15), (22) are sufficient for convergence of u_n. If condition (22) holds, then the error estimate (16) is of the form

$$\|u_n - u_*\|_E \le (1+M) \min_{v_n \in E_n} \|u_* - v_n\|_E + \breve{\kappa}_n \delta, \tag{23}$$

this together with (15) guarantees convergence (17) for $\delta = 0$ and with parameter choice (20) also for $\delta \to 0$.

To show necessity of conditions (15), (22) for convergence of u_n, note that convergence (21) for every f^δ with $\|f^\delta - f\| \le \delta$ implies convergence (17) for f (i.e. convergence (17) for $\delta = 0$). Let $\delta = 0$ and $u_n \to u_*$ for all $u_* \in E$ as $n \to \infty$. Then (15) holds. But in case $u_n = A_n^{-1}Q_nAu_* \to u_*$ we have that $A_n^{-1}Q_nA \to I$ pointwise on E. By the uniform boundedness principle (Banach–Steinhaus theorem) this implies that $A_n^{-1}Q_nA$ must be uniformly bounded, which is condition (22). □

Remark 2. The boundedness property (22) holds, if uniformly bounded operators $\{P_n : E \to E_n, n \in N\}$ exist, satisfying

$$\breve{\kappa}_n \|A(I - P_n)\| \le M. \tag{24}$$

Namely condition (22) is equivalent to the condition

$$\|A_n^{-1}Q_nA(I - P_n)\| \le M', \tag{25}$$

while $A_n^{-1}Q_nA(I - P_n) = A_n^{-1}Q_nA - A_n^{-1}Q_nAP_n$ and the operator $A_n^{-1}Q_nAP_n = P_n$ is bounded. If (24) holds then using equality $\breve{\kappa}_n = \|A_n^{-1}Q_n\|$ we get (25).

For the convergence analysis in case of exact data we can choose different image spaces, particularly such that the equation becomes well-posed. But for noisy data the image space is determined by the data.

The following theorem (about the case of the exact data) shows, that convergence for one equation implies convergence also for certain other equations.

Theorem 3. *Let the operator A be injective. Let conditions (11), (12), (18) hold for $n \geq n_0$. Let the operator $A : E \to F$ have the form $A = S + K$, where $S : E \to W \subset F$ is invertible, W is a Banach space with continuous imbedding and $K : E \to W$ is compact. Let the operator $S_n := Q_n S|_{E_n} : E_n \to Z_n^*$ be invertible and $\|S_n^{-1} Q_n S\| \leq M$ for some constant M. Then the projection equation $Q_n A u_n = Q_n f$ has for n large enough a unique solution $u_n \in E_n$, and $u_n \to u_*$ as $n \to \infty$.*

Proof. From compactness of K follows the compactness of operator $S^{-1}K$. Denote $S_n = Q_n S|_{E_n}$. Since $S^{-1} : W \to E$ is bounded, the pointwise convergence $S_n^{-1} Q_n S \to I$ on W as $n \to \infty$ implies the pointwise convergence $S_n^{-1} Q_n \to S^{-1}$ as $n \to \infty$. From the pointwise convergence $S_n^{-1} Q_n \to S^{-1}$ and the compactness of K follows the norm convergence

$$\|(I + S_n^{-1} Q_n K) - (I + S^{-1} K)\| \to 0 \text{ as } n \to \infty.$$

Therefore the inverse operator $[I + S_n^{-1} Q_n K]^{-1} : E_n \to E_n$ exists and is uniformly bounded for large n. Due to equality $Q_n A = Q_n S[I + (Q_n S)^{-1} Q_n K]$ the operator $Q_n A$ on E_n is invertible for large n with the inverse

$$(Q_n A)^{-1} = [I + (Q_n S)^{-1} Q_n K]^{-1} (Q_n S)^{-1}.$$

The equality

$$(Q_n A)^{-1} Q_n A = [I + (Q_n S)^{-1} Q_n K]^{-1} (Q_n S)^{-1} Q_n S (I + S^{-1} K)$$

allows to estimate

$$\|(Q_n A)^{-1} Q_n A\| \leq \|[I + (Q_n S)^{-1} Q_n K]^{-1}\| M \|I + S^{-1} K\| =: M_K.$$

This estimate may be rewritten in the form $\|A_n^{-1} Q_n A\| \leq M_K$, where the constant M_K depends on the operator K. Therefore condition (22) is satisfied and Theorem 2 guarantees convergence. □

For considering the influence of the noisy data, the behaviour of the quantities $\breve{\kappa}_n$ is essential. For estimating these quantities we introduce operators $\Pi_n : Z_n^* \to F$ such that the equality $Q_n \Pi_n Q_n = Q_n$ holds. Then the operator $\Pi_n Q_n$ is a projector in F. Let $F_n = \mathscr{R}(\Pi_n)$. We assume that $F_n \subset W$ and let $W_n = F_n$, equipped with the norm of W. Let I_n be the identity operator, considered as acting from F_n to W_n.

Theorem 4. *Let conditions (11), (12), (22) hold for $n \geq n_0$. Let the operator $A : E \to W$ be invertible. Assume the projectors $\Pi_n Q_n$ are uniformly bounded in F. Then*

$$\breve{\kappa}_n \leq C \|I_n\|_{F_n \to W_n}, \qquad n \geq n_0. \tag{26}$$

Proof. We have

$$\breve{\kappa}_n = \|A_n^{-1}Q_n\|_{F\to E_n} = \|A_n^{-1}Q_n I_n \Pi_n Q_n\|_{F\to E_n} = \|A_n^{-1}Q_n A A^{-1} I_n \Pi_n Q_n\|_{F\to E_n} \leq$$
$$\leq \|A_n^{-1}Q_n A\|_{E\to E}\|A^{-1}\|_{W\to E}\|I_n\|_{F_n\to W_n}\|\Pi_n Q_n\|_{F\to F_n}.$$

This implies (26), since the other multipliers besides $\|I_n\|_{F_n\to W_n}$ are bounded. □

We point out that the choice of operators Π_n is quite arbitrary and is not determined by the method itself. For example, in collocation methods $\Pi_n Q_n$ should be an interpolation projector, but it can be interpolation by splines, or polynomial interpolation or trigonometric interpolation or maybe something else, which may suit the particular problem. The only conditions are that the result is smooth enough (it must belong to the space W) and $\Pi_n Q_n$ are uniformly bounded.

Estimates for $\|I_n\|_{F_n\to W_n}$ can be found using inverse properties of approximation subspaces (estimating elements of F_n via their norm in W_n). Splines are often useful here, because their inverse properties (estimates of the derivatives in terms of the splines themselves) are well known. Estimates for operators $\|A(I-P_n)\|$ in condition (24) can be derived from the approximation properties of the approximation subspaces. Often the norm $\|(I-P_n^*)A^*\| = \|A(I-P_n)\|$ is easier to estimate.

2.2 Convergence with a posteriori choice of n – the discrepancy principle

For the discrepancy principle, in previous works the assumption (14) about uniform boundedness of τ_n was required. For collocation methods this is the uniform boundedness of the interpolation projector onto the subspace $AE_n \subset F$. If $F = C^m$, (14) does not hold in general. In the next two theorems we consider two versions of the discrepancy principle, condition (14) is assumed only in the first version.

Theorem 5. *Let the assumptions of Lemma 1 be satisfied for $n \geq n_0$, and let u_n be defined by the projection method (3). Let the convergence*

$$\breve{\kappa}_{n+1} dist(f, AE_n) \to 0 \text{ as } n \to \infty \tag{27}$$

holds. We also assume that there exists a sequence of approximations $(\hat{u}_n)_{n\in\mathbb{N}}$, $\hat{u}_n \in E_n$, satisfying (18) and (19). Let condition (14) holds. Let $b > \tau + 1$ be fixed and for $\delta > 0$, let $n = n_{DP}(\delta)$ be the first index such that

$$\|Au_n - f^\delta\|_F \leq b\delta. \tag{28}$$

Then $n_{DP}(\delta)$ is finite and

$$\|u_{n_{DP}(\delta)} - u_*\|_E \to 0 \text{ as } \delta \to 0. \tag{29}$$

Proof. For any n let $v_n \in E_n$ be such that $\|f^\delta - Av_n\| = \mathrm{dist}(f^\delta, AE_n)$. We have

$$\|Au_n - f^\delta\|_F \le \|A(u_n - v_n)\|_F + \|Av_n - f^\delta\|_F \le$$

$$\le \tau_n \|Q_n A(u_n - v_n)\| + \|Av_n - f^\delta\|_F = \tau_n \|Q_n(f^\delta - Av_n)\| + \|Av_n - f^\delta\|_F \le$$

$$\le (\tau_n + 1)\,\mathrm{dist}(f^\delta, AE_n) \le (\tau_n + 1)\,(\delta + \mathrm{dist}(f, AE_n)). \tag{30}$$

This inequality together with (14) and relation

$$\mathrm{dist}(f, AE_n) \le \|A(u_* - \hat{u}_n)\| \to 0 \quad \text{ as } n \to \infty \tag{31}$$

imply that n_{DP} is finite.

If for some $\delta_k \to 0\,(k \to \infty)$ the discrepancy principle gives $n_{DP}(\delta_k) \le N$ with $N \ge 0$, then the sequence $u_{n_{DP}(\delta_k)}$ lies in a finite-dimensional subspace – the linear hull of E_n, $n = 1, \dots, N$. Since

$$\|Au_{n_{DP}(\delta_k)} - f^{\delta_k}\|_F \le b\delta_k, \tag{32}$$

then $Au_{n_{DP}(\delta_k)} \to f$ as $k \to \infty$. This implies convergence $u_{n_{DP}(\delta_k)} \to u_*$ as $k \to \infty$, since the operator A has the bounded inverse on finite-dimensional subspaces.

Consider now the general case $n_{DP}(\delta) \to \infty$ as $\delta \to 0$. Let $m = n_{DP}(\delta) - 1 \ge 0$. For $n = m$ the inequality (28) does not hold, and (30) with (14) gives

$$b\delta < \|Au_m - f^\delta\|_F \le (\tau + 1)(\delta + \mathrm{dist}(f, AE_m)), \tag{33}$$

therefore also

$$\frac{(b - 1 - \tau)\delta}{\tau + 1} < \mathrm{dist}(f, AE_m). \tag{34}$$

The convergence (27) implies

$$\breve{\kappa}_{n_{DP}}\delta < \frac{\tau + 1}{b - 1 - \tau}\breve{\kappa}_{n_{DP}}\mathrm{dist}(f, AE_{n_{DP}-1}) \to 0 \text{ as } n_{DP} \to \infty. \tag{35}$$

Therefore the second term in estimate (16) converges as $n = n_{DP} \to \infty$. Convergence of the first term there follows as in the proof of Theorem 1, using $v_n = \hat{u}_n$, $n = n_{DP}(\delta)$ and assumptions (18), (19). □

Theorem 6. *Let the assumptions of Theorem 5 be satisfied without requirement (14). Let the sequence*

$$b_n > (1 + \tau_n)(1 + \varepsilon) \tag{36}$$

be fixed with some fixed $\varepsilon > 0$ and $n = n_{DP}(\delta)$ be chosen as the first index such that

$$\|Au_n - f^\delta\|_F \le b_n\delta. \tag{37}$$

Then $n_{DP}(\delta)$ is finite and the convergence (29) holds.

Proof. The proof is similar to the proof of the previous theorem. Condition (36) gives the inequality $(\tau_n + 1)\delta \le b_n\delta - \varepsilon(\tau_n + 1)\delta$, and the estimate (30) can be continued as follows:

$$\|Au_n - f^\delta\|_F \le b_n\delta + (\tau_n+1)(\operatorname{dist}(f, AE_n) - \varepsilon\delta).$$

Due to convergence (31) the second summand here will be negative for sufficiently large n, therefore $n_{DP}(\delta)$ will be finite. If for some $\delta_k \to 0\,(k\to\infty)$ the discrepancy principle gives $n_{DP}(\delta_k) \le N$, the proof of convergence $u_{n_{DP}(\delta_k)} \to u_*$ as $k\to\infty$ is the same as in the previous theorem with the exception, that the inequality $\|Au_{n_{DP}(\delta_k)} - f^{\delta_k}\|_F \le b_N\delta_k$ is used instead of (32). The proof of convergence (29) in case $n_{DP}(\delta)\to\infty$ as $\delta\to 0$ is the same as in the previous theorem, only in the inequalities (33), (34), (35) the quantities b, τ and $\dfrac{\tau+1}{b-1-\tau}$ are replaced by b_m, τ_m and $\varepsilon^{-1} < \dfrac{\tau_m+1}{b_m-1-\tau_m}$, respectively. □

3 The least squares method

In the least squares method (4) we use the condition

$$\mathscr{N}(A)\cap E_n = \{0\} \tag{38}$$

instead of the requirement of the injectivity of the operator A. In [16] the following result is proved.

Theorem 7. *Let condition* (38) *be satisfied for all $n\in\mathbb{N}$. Then an approximation u_n according to the least squares method* (4) *exists and the error estimate*

$$\|u_n - u_*\| \le \inf_{v_n\in E_n}\{\|u_*-v_n\|_E + 2\kappa_n\|Au_* - Av_n\|_F\} + 2\kappa_n\delta$$

holds. If there exists a sequence of approximations $(\hat u_n)_{n\in\mathbb{N}}$, $\hat u_n\in E_n$, satisfying (18) *and*

$$\kappa_n\|A(u_*-\hat u_n)\|_F \to 0 \text{ as } n\to\infty, \tag{39}$$

then we have in case of exact data convergence $\|u_n-u_*\|_E\to 0$ as $n\to\infty$, and in case of noisy data with the choice of $n=n(\delta)$ according to

$$n(\delta)\to\infty \text{ and } \kappa_{n(\delta)}\delta\to 0 \text{ as } \delta\to 0$$

convergence

$$\|u_{n(\delta)}-u_*\|_E \to 0 \text{ as } \delta\to 0. \tag{40}$$

If in addition to convergences (18), (39) *also $\kappa_{n+1}\|A(u_*-\hat u_n)\|_F\to 0$ as $n\to\infty$ holds, then convergence* (40) *holds also with the choice of $n(\delta)$ by the discrepancy principle: for fixed $b>1$ choose $n(\delta)$ as the first index such that $\|Au_n - f^\delta\| < b\delta$.*

The discrepancy principle fits better to the least squares method than to other projection methods in the sense that there is no need to calculate or estimate the quantities τ, τ_n which may be a hard task.

4 Application: collocation method for Volterra integral equations

We consider collocation method for Volterra integral equations. In the first two examples these equations are cordial integral equations studied in [18, 28, 29, 30, 31]. We give properties of these equations in Section 4.1 and consider the collocation method in Section 4.2.

4.1 Cordial integral equations

Consider cordial integral equations of the first kind

$$\int_0^t \frac{1}{t} a(t,s)\phi(\frac{s}{t})u(s)ds = f(t), \quad 0 \le t \le T, \tag{41}$$

where $\phi \in L^1(0,1)$ is called the core of the cordial integral operator, and a, f are given smooth enough functions. Define the cordial integral operators

$$(V_\phi u)(t) = \int_0^t \frac{1}{t}\phi(\frac{s}{t})u(s)ds, \quad (V_{\phi,a}u)(t) = \int_0^t \frac{1}{t} a(t,s)\phi(\frac{s}{t})u(s)ds.$$

Denote $\Delta_T = \{(s,t) : t \in [0,T], s \in [0,t]\}$. The following results are proven in [28, 29, 30, 31].

Theorem 8. *Let $\phi \in L^1(0,1)$, $a \in C^m(\Delta_T)$. Then $V_{\phi,a} \in \mathscr{L}(C^m[0,T])$ and*

$$\|V_{\phi,a}\|_{C^m[0,T]} \le C\|\phi\|_{L^1(0,1)}\|a\|_{C^m(\Delta_T)}.$$

Theorem 9. *Let $\phi \in L^1(0,1)$ and let $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > 0$. Then t^λ is an eigenfunction of V_ϕ in $C[0,T]$, and the corresponding eigenvalue is $\hat{\phi}(\lambda) = \int_0^1 \phi(x)x^\lambda dx$. If $\operatorname{Re}\lambda > m$, then the eigenfunction belongs to $C^m[0,T]$.*

Theorem 10. *Let $\phi \in L^1(0,1)$, $a \in C^m(\Delta_T)$. Then the spectrum of $V_{\phi,a}$ in $C^m[0,T]$ is given by $\sigma_m(V_{\phi,a}) = \{0\} \cup \{a(0,0)\hat{\phi}(k),\ k = 0,\dots,m\} \cup \{a(0,0)\hat{\phi}(\lambda),\ \operatorname{Re}\lambda > \mathrm{m}\}$.*

Theorem 11. *Let $\phi \in L^1(0,1)$, $x(1-x)\phi'(x) \in L^1(0,1)$, $\int_0^1 \phi(x)dx > 0$ and there exists $\beta < 1$ such that $(x^\beta \phi(x))' \ge 0$ for $x \in (0,1)$. Assume also that $a \in C^{m+1}(\Delta_T)$ and $a(t,t) \ne 0$. Then $V_{\phi,a}$ is injective in $C[0,T]$, $C^{m+1}[0,T] \subset V_{\phi,a}(C^m[0,T]) \subset C^m[0,T]$, and $V_{\phi,a}^{-1} \in \mathscr{L}(C^{m+1}[0,T], C^m[0,T])$.*

Corollary 1. *Let the assumptions of Theorem 11 be satisfied and let $f \in C^{m+1}[0,T]$ be given. Then the equation (41) is uniquely solvable in $C[0,T]$ and its solution is in $C^m[0,T]$.*

4.2 Polynomial collocation method for cordial integral equations, numerical results

According to Theorem 9, functions $t^k, k \in \mathbb{N}$ are eigenfunctions of the cordial integral operator V_ϕ, therefore the polynomial collocation method is well adapted for these equations. We look for solutions in the form $u_n(s) = \sum_{j=0}^n c_j s^j$. In the collocation method we choose the collocation points $t_k \in [0,T]$, $k = 0,\dots,n$ and find c_k, $k = 0,\dots,n$ from the collocation equations

$$\sum_{j=0}^n c_j \int_0^{t_k} \frac{1}{t_k} a(t_k,s)\phi(\frac{s}{t_k}) s^j ds = f(t_k), \quad k = 0,\dots,n.$$

To set up the system, one has to calculate exactly or "well enough" the integrals

$$\int_0^t \frac{1}{t_k} a(t_k,s)\phi(\frac{s}{t_k}) s^j ds.$$

For theoretical results it is convenient to use the basis $\{s^j\}$ for polynomials; for practical calculations though, this results in very badly conditioned systems. So for larger N one has to use a better basis, for example the (scaled) Chebyshev polynomials $T_p(t) = \cos\left(p \arccos\left(\frac{2t}{T} - 1\right)\right)$. In fact, it may be simpler to make first the change of variables $t = \frac{T}{2}(1 - \cos y)$ and then work with trigonometric polynomials in y instead.

In the following examples 1, 2, $E = F = C[0,T]$, E_n is the space of polynomials of order up to n and Z_n is the linear span of δ-functions with supports t_k, $k = 0,\dots,n$. Let $a(t,s) \equiv 1$. Then $V_\phi : E_n \to E_n$ and τ_n is simply the norm of the interpolation projector from C to C with the interpolation nodes t_k, $k = 0,\dots,n$. If t_k are the Chebyshev nodes, then $\tau_n \approx \frac{2}{\pi}\ln(n+1) + 1$.

In Examples 1, 2 certain noise levels were chosen and the noise was generated by random numbers with uniform distribution at the collocation nodes, and on nine times denser mesh for calculating the discrepancy. We also found the optimal number n_{opt} and the corresponding error $e_{opt} = \min_{n\in\mathbb{N}} \|u_n - u_*\|_E = \|u_{n_{opt}} - u_*\|_E$. The discrepancy principle was used for finding proper $n = n(\delta)$. The condition (14) is not satisfied in Examples 1,2. According to the discrepancy principle from Theorem 6 we found the first $n = n_{DP}$ satisfying the inequality $\|Au_n - f^\delta\|_F \le b_n\delta$ with $b_n = 1.001(1 + \tau_n)$. We denote the corresponding error by $e_{DP} = \|u_{n_{DP}} - u_*\|$. The optimal errors and the errors obtained by using the discrepancy principle are presented in the following Tables 1, 2. In these tables also $b_{n_{DP}}$ are presented.

Example 1. Consider the cordial integral equation (here $\phi(x) = \frac{1}{\sqrt{x}}$)

$$\int_0^t \frac{u(s)ds}{\sqrt{st}} = \frac{1}{t^2+1}, \quad t \in [0,T] \tag{42}$$

with exact solution $u(s) = \frac{1-3s^2}{2(s^2+1)^2}$. For this equation κ_n can be estimated using Markoff's inequality, by Cn^2. Since the right-hand side of the equation is analytic, $\mathrm{dist}(f, AE_n)$ converges to zero exponentially, hence the assumptions of Theorem 6 are satisfied.

We took $T = 10$ and used noisy data with noise levels $\delta = 10^{-4}, 10^{-6}, \ldots, 10^{-14}$. The number of collocation nodes was 10, 15, 20, ..., 110.

δ	e_{opt}	n_{opt}	e_{DP}	n_{DP}	$b_{n_{DP}}$
10^{-4}	$6 \cdot 10^{-2}$	25	$8 \cdot 10^{-2}$	20	3.94
10^{-6}	$1.01 \cdot 10^{-3}$	40	$2.4 \cdot 10^{-3}$	30	4.19
10^{-8}	$1.51 \cdot 10^{-5}$	40	$1.51 \cdot 10^{-5}$	40	4.36
10^{-10}	$1.8 \cdot 10^{-7}$	50	$1.8 \cdot 10^{-7}$	50	4.56
10^{-12}	$4.69 \cdot 10^{-9}$	75	$9.58 \cdot 10^{-9}$	60	4.62
10^{-14}	$7.04 \cdot 10^{-11}$	105	$7.57 \cdot 10^{-11}$	70	4.71

Table 1 Optimal errors with the corresponding n_{opt} and errors obtained by using the discrepancy principle with $b_{n_{DP}}$ for equation (42).

Example 2. Consider the equation

$$\int_0^t \frac{u(s)ds}{\sqrt{st}} = t^{3/2}(2-t)^{5/2}, \quad t \in [0,2]. \tag{43}$$

The exact solution is $u(s) = 2s^{3/2}(2-s)^{5/2} - \frac{5}{2}s^{5/2}(2-s)^{3/2}$. Since the integral operator is the same as in Example 1, κ_n is the same. The distance $\mathrm{dist}(f, AE_n)$ can be estimated by Cn^{-3}, hence the assumptions of Theorem 6 are satisfied.

We used noisy data with noise levels $\delta = 10^{-3}, 10^{-4}, \ldots, 10^{-7}$. The number of collocation nodes was 10, 20, 30, ..., 300.

δ	e_{opt}	n_{opt}	e_{DP}	n_{DP}	$b_{n_{DP}}$
10^{-3}	$1.5 \cdot 10^{-1}$	10	$1.5 \cdot 10^{-1}$	10	3.53
10^{-4}	$5 \cdot 10^{-2}$	40	$1.1 \cdot 10^{-1}$	30	3.94
10^{-5}	$5.24 \cdot 10^{-3}$	20	$2 \cdot 10^{-2}$	50	4.5
10^{-6}	$6.13 \cdot 10^{-4}$	40	$5.16 \cdot 10^{-3}$	100	4.94
10^{-7}	$9.17 \cdot 10^{-5}$	90	$5.77 \cdot 10^{-3}$	230	5.46

Table 2 Optimal errors with the corresponding n_{opt} and errors obtained by using the discrepancy principle with $b_{n_{DP}}$ for equation (43).

4.3 Spline-collocation for Volterra integral equation, numerical results

We consider a Volterra integral equation of the first kind

$$(Au)(t) := \int_0^t K(t,s)u(s)\,ds = f(t),\ t \in [0,1] \tag{44}$$

with the operator $A \in L(L^p(0,1), C[0,1]),\ 1 \le p \le \infty$. The approximation space is $E_n = S_{k-1}^{(-1)}(I_\Delta)$, the space of discontinuous piecewise polynomials of order $k-1$ with mesh Δ. In the collocation method we find u_n from the spline space E_n such that

$$Au_n(t_{i,j}) = f^\delta(t_{i,j}), \quad i = 1,\ldots,n,\ j = 1,\ldots,k$$

where $t_{i,j} = (i-1+c_j)h \in [0,1]$, $i = 1,\ldots,n,\ j = 1,\ldots,k$ are collocation nodes and $0 < c_1 < \ldots < c_k \le 1$ are collocation parameters whose choice is essential.

Example 3. Consider the equation

$$Au(t) = \int_0^t u(s)ds = \frac{t^q}{q}, \quad t \in [0,1], \quad q \in \{3/2, 5/2\} \tag{45}$$

with operator $A : L^1(0,1) \to C[0,1]$. The exact solution is $u(s) = s^{q-1}$. We used for E_n the space of discontinuous linear splines with uniform mesh ih, $i = 0,\ldots,n$, where $h = 1/n$. The collocation points are $t_{i1} = (i-1+c)h$, $t_{i2} = ih$, $c \in (0,1)$. For this problem $\breve{\kappa}_n$ can be estimated using Theorem 4. Here we can take for F_n the space of continuous linear splines and the inverse property of these splines gives

$$\forall w_n \in F_n, \quad \|w_n'\| \le Cn\|w_n\|,$$

hence $\breve{\kappa}_n \le Cn$. The distance $\mathrm{dist}(f, AE_n)$ can be estimated by Cn^{-q}.

It can be shown that here

$$\tau = \begin{cases} 1 + \frac{c^2}{2(1-c)}, & \text{if } c \ge \frac{1}{2}, \\ 1 + \frac{(1-c)^2}{2c} & \text{if } c \le \frac{1}{2}. \end{cases}$$

The quantity τ is minimal for $c = \frac{1}{2}$, then $\tau = 1.25$. In this example $\tau_n = \tau$ holds, i.e. τ_n does not depend on n. We used $c = \frac{1}{2}$ and for satisfying the condition $b > \tau + 1$ in Theorem 5 we actually took $b = 1.01 + \tau = 2.26$ for the discrepancy principle.

The noisy data were generated by the formula $f^\delta(t_{i,j}) = f(t_{i,j}) + \delta\theta_{i,j}$, where $\delta = 10^{-m}, m \in \{2,\ldots,7\}$ and $\theta_{i,j}$ are random numbers with normal distribution, normed after being generated: $\max_{i,j}|\theta_{i,j}| = 1$.

We can conclude that for these model problems the discrepancy principle gave reasonable results.

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δ	e_{opt}	n_{opt}	e_{DP}	n_{DP}	e_{opt}	n_{opt}	e_{DP}	n_{DP}
10^{-1}	$2.5\cdot10^{-1}$	1	$2.5\cdot10^{-1}$	1	$2.9\cdot10^{-1}$	1	$2.9\cdot10^{-1}$	1
10^{-2}	$6.8\cdot10^{-2}$	2	$6.8\cdot10^{-2}$	2	$5.4\cdot10^{-2}$	2	$5.4\cdot10^{-2}$	2
10^{-3}	$1.3\cdot10^{-2}$	8	$1.8\cdot10^{-2}$	5	$9\cdot10^{-3}$	6	$1.1\cdot10^{-2}$	5
10^{-4}	$3.2\cdot10^{-3}$	24	$3.3\cdot10^{-3}$	20	$1.7\cdot10^{-3}$	15	$3\cdot10^{-3}$	8
10^{-5}	$7.6\cdot10^{-4}$	72	$8.4\cdot10^{-4}$	86	$3.5\cdot10^{-4}$	32	$6.2\cdot10^{-4}$	18
10^{-6}	$1.9\cdot10^{-4}$	128	$3.3\cdot10^{-4}$	512	$6.8\cdot10^{-5}$	72	$9.9\cdot10^{-5}$	46
10^{-7}	$4.5\cdot10^{-5}$	512	$1.2\cdot10^{-4}$	2048	$1.5\cdot10^{-5}$	128	$1.5\cdot10^{-5}$	128

Table 3 Optimal errors and errors obtained by using the discrepancy principle for equation (45); left with $q = 3/2$ and right with $q = 5/2$.

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