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Contents

Monotonicity of error and choice of the stopping index in iterative regularization methods

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ABSTRACT. We consider linear ill-posed problems $Au = f$ with given noisy data f_δ satisfying $\|f_\delta - f\| \leq \delta$ with known δ. For solving $Au = f$ iterative methods $u_n = u_{n-1} - g(A^*A)A^*(Au_{n-1} - f_\delta)$, $n = 1, 2, \ldots$ with $g(\lambda) \in (0, 2/\lambda)$ for $0 \leq \lambda \leq \|A^*A\|$ are considered, the choice of stopping index n is discussed. A rule (what we call the monotone error rule, ME-rule) is proposed for finding n_{ME} as maximal index such that the monotonicity property $\|u_n - u_*\| < \|u_{n-1} - u_*\|$ for $n \leq n_{ME}$ is guaranteed, where u_* is nearest to u_0 solution of $Au = f$. For explicit iteration scheme ($g(\lambda) = \text{const} > 0$) and implicit iteration scheme ($g(\lambda) = (\lambda+\rho)^{-1}$ with $\rho = \text{const} > 0$) the relation $n_D - 1 \leq n_{ME} \leq n_D$ is proved, where n_D is the first n with $\|Au_n - f_\delta\| \leq \delta$ (discrepancy principle). Comparisons with the rule for choice of n by Engl and Gfrerer [2] are made.

1. Introduction

Consider linear ill-posed problem

$$Au = f\,, \tag{1}$$

where $A \in \mathcal{L}(H, F)$is a linear bounded operator with nonclosed range $\mathcal{R}(A)$ of A and H, F are infinite dimensional real Hilbert spaces with corresponding inner products $(\cdot, \cdot)$ and norms $\|\cdot\|$, respectively. Assume that instead of $f \in \mathcal{R}(A)$ only element $f_\delta \in F$ with $\|f_\delta - f\| \leq \delta$ and known noise level δ is available.

For solving ill-posed problems regularization methods are necessary (see [12–15, 7, 3]). Many regularization methods can be generated by Borel measurable functions $g_r(\lambda) : [0, a] \to \mathbb{R}$ ($r \geq 0$, $\|A\|^2 \leq a$), satisfying conditions

$$\sup_{0 \leq \lambda \leq a} |g_r(\lambda)| \leq \overline{\gamma} r \qquad (r \geq 0)\,, \tag{2}$$

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$$\sup_{0\le\lambda\le a} \lambda^p|1-\lambda g_r(\lambda)| \le \gamma_p r^{-p} \qquad (r\ge 0,\ 0\le p\le p_0). \tag{3}$$

Here $\overline{\gamma}$, γ_p and p_0 are positive constants ($\overline{\gamma}$, γ_p are smallest constants, for which (2), (3) holds).

Let $u_0 \in H$ be an initial approximation. The nearest to u_0 solution u_* of (1) may be approximated by

$$u_r = u_0 + g_r(A^*A)A^*(f_\delta - Au_0)\,. \tag{4}$$

The class of regularization methods (4) with conditions (2), (3) was introduced in [12, 13] and later often used.

Examples of methods of form (4) are m-iterated Tikhonov method

$$u_{0,r} = u_0\,, \quad u_{k,r} = (r^{-1}I + A^*A)^{-1}(r^{-1}u_{k-1,r} + A^*f_\delta)\,; \quad u_r = u_{m,r} \\ k = 1,\dots,m \tag{5}$$

(here $g_r(\lambda) = \lambda^{-1}(1-(1+r\lambda)^{-m})$, $p_0 = m$; case $m = 1$ gives the ordinary Tikhonov method), the method of Cauchy problem

$$r = t\,, \quad u_r = u(t)\,, \quad \frac{du}{dt} + A^*Au = A^*f_\delta\,, \quad u(0) = u_0 \tag{6}$$

(here $g_r(\lambda) = (1-e^{r\lambda})/\lambda$, $p_0 = \infty$) and others (see [12–15]).

In this paper we restrict ourselves mainly to the iterative methods in the form

$$u_{n+1} = u_n - g(A^*A)A^*(Au_n - f_\delta)\,, \qquad n = 0,1,\dots\,, \tag{7}$$

where $g : [0,a] \to \mathbb{R}$ is a continuous function satisfying

$$0 < g(\lambda) < \frac{2}{\lambda} \quad \text{for} \quad 0 \le \lambda \le a\,. \tag{8}$$

These iterative methods belong to the class of methods (4) with $r = n$, $g_r(\lambda) = \lambda^{-1}(1-(1-\lambda g(\lambda))^r)$, $p_0 = \infty$. Special cases of iterative methods (7) are the explicit iteration scheme

$$u_{n+1} = u_n - \mu A^*(Au_n - f_\delta)\,, \quad n = 0,1,\dots \quad (\mu = \text{const} \in (0, 2\|A\|^{-2})) \tag{9}$$

with constant function $g(\lambda) = \mu$ and the implicit iteration scheme

$$\rho u_{n+1} + A^*Au_{n+1} = \rho u_n + A^*f_\delta\,, \quad n = 0,1,\dots \quad (\rho = \text{const} > 0) \tag{10}$$

with function $g(\lambda) = (\rho+\lambda)^{-1}$.

It is easy to verify that in all considered special cases of method (4) the inequality

$$|1 - \lambda g_{r_1}(\lambda)| \leq |1 - \lambda g_{r_2}(\lambda)| \qquad (0 \leq \lambda \leq a\,,\ 0 < r_2 \leq r_1) \tag{11}$$

hold (the function $|1-\lambda g_r(\lambda)|$ is monotone decreasing with respect r) and this leads in case of exact data ($\delta = 0$) to monotone convergence $\|u_r - u_*\| \to 0$ ($r \to \infty$), where u_* is nearest to u_0 solution of (1) (see [12, 13, 15]). In case of noisy data ($\delta > 0$) the monotone decrease of error $\|u_r - u_*\|$ with respect of r can be proved only for small r (for $r \in [0, r_*]$ with some r_*), typically $\|u_r - u_*\|$ diverge for $r \to \infty$. Therefore the question about a proper choice of the regularization parameter r is important. We propose to take for the value of regularization parameter the boundary point r_* where known monotonical convergence of u_r for exact data goes over to uncertainty due to noise. We call this rule the monotone error rule (ME-rule).

As known (see e.g. [15, 3]), the a priori choice $r = r(\delta)$ by conditions $r(\delta) \to \infty$ ($\delta \to 0$), $r(\delta)\delta^2 \to 0$ ($\delta \to 0$) guarantee convergence $\|u_r - u_*\| \to 0$ ($\delta \to 0$). This rule is too universal and for choice of r by regularization a concrete problem it is better to take into account concrete data.

In this paper we consider a posteriori parameter choice rules in regularization methods, mainly in iterative methods (7), (9), (10) with number of iterations n as regularization parameter. We propose a new rule, the ME-rule for choice of stopping index n in iterative methods. We compare this rule with other rules for choice of n and give recommendations for choice of free constants in these rules. As shown, for recommended constants considered rules are almost equivalent.

2. A posteriori parameter choice rules

a) Well-known rules. Usually the regularization parameter $r(\delta)$ is chosen in a posteriori way from rule $d(r) \approx \delta$, where $d(r)$ is a fixed functional. More exactly, in case of continuous dependence of $d(r)$ from r parameter $r(\delta)$ is found as a solution of nonlinear equation $d(r) = \delta$. In iterative methods the regularization parameter $n(\delta)$ is chosen as first $n = 0, 1, 2, \ldots$, for which $d(n) \leq \delta$ holds. The functional $d(r)$ has in well-known parameter choice rules the following form:

- discrepancy principle [8]:

$$d(r) = d_D(r) \equiv \|Au_r - f_\delta\|/b\,, \qquad b = \text{const} > 1 \tag{12}$$

- rule of Raus [9]:

$$d(r) = d_R(r) \equiv \|(I - AA^* g_r(AA^*))^{1/(2p_0)}(Au_r - f_\delta)\|/b\,, \quad b = \text{const} > 1 \tag{13}$$

- rule of Engl and Gfrerer [2] for continuous methods (with differentiable $g_r(\lambda)$):

$$d(r) = d_{EG}(r) \equiv \gamma^{-1/2}\Big(Au_r - f_\delta, \tfrac{dg_r(AA^*)}{dr}(Au_0 - f_\delta)\Big)^{1/2}, \quad \gamma = \text{const} > \overline{\gamma} \text{ (see (2))} \tag{14}$$

- rule of Engl and Gfrerer [2] for iterative methods:

$$d(n) = d_{EG}(n) \equiv \kappa^{-1/2}\big(A(u_n + u_{n+1})/2 - f_\delta, g(AA^*)(Au_n - f_\delta)\big)^{1/2}, \quad \kappa = \text{const} > \overline{\kappa} \equiv \sup\{g(\lambda)\,|\,0 \le \lambda \le a\}\,. \tag{15}$$

Note that formulations of parameter choice rules in [2] were given in other terms. Note also that for m-iterated Tikhonov method rules of Raus [9], Gfrerer [4], Engl and Gfrerer [2] coincide, parameter $r = r_{RG}$ is found from equation

$$d_{RG}(r) \equiv (Au_r - f_\delta, Au_{m+1,r} - f_\delta)/b = \delta\,, \tag{16}$$

where $u_{m+1,r}$ is approximation in $(m+1)$-iterated Tikhonov method.

As stated in [12, 14, 15, 4, 2], equalities $b = 1$, $\kappa = \overline{\kappa}$ are allowed for m-iterated Tikhonov method, equality $b = 1$ also for method (6) and (7) with

$$(\lambda + \kappa^{-1})^{-1} \le g(\lambda) \le \lambda^{-1}. \tag{17}$$

Note that (17) requires $\mu = \text{const} \in (0, \|A^{-2}\|]$ in (9). We consider case $b = 1$, $\kappa = \overline{\kappa}$ in Section 5.

Overview of various parameter choice rules is given in [3, 7, 11].

b) The monotone error rule (ME-rule). We assume that (11) holds. Then in case of exact data ($\delta = 0$) $u_r \to u_*$ ($r \to \infty$) monotonically. In case of noisy data we propose for choice of the parameter r the following ME-rule.

ME-rule. Choose the regularization parameter $r_{ME} = r_{ME}(\delta)$ as the largest r-value for which we are able to prove (under the assumption $\|f_\delta - f\| \le \delta$) that the error $\|u_r - u_*\|$ is monotonically decreasing for $r \in [0, r_{ME}]$.

In case of continuous regularization methods with differentiable function $g_r(\lambda)$ with respect r it means

$$\frac{d}{dr}\|u_r - u_*\|^2 \leq 0 \qquad (\forall r \in (0, r_{ME}]\,), \tag{18}$$

in case of iterative methods it means

$$\|u_n - u_*\| < \|u_{n-1} - u_*\| \qquad (\forall n \leq n_{ME}\,). \tag{19}$$

For continuous regularization methods with continuously differentiable function $g_r(\lambda)$ with respect to r it was shown in [5], that monotonicity property (18) is guaranteed by choice of $r = r_{ME}(\delta)$ from the equation

$$d_{ME}(r) \equiv \frac{(Au_r - f_\delta\,,\ \frac{d}{dr}g_r(AA^*)(Au_0 - f_\delta))}{\|\frac{d}{dr}g_r(AA^*)(Au_0 - f_\delta)\|} = \delta\,, \tag{20}$$

if this function $d_{ME}(r)$ is monotonically decreasing. For m-iterated Tikhonov method (5) corresponding to (20) equation has the form

$$d_{ME}(r) \equiv \frac{(Au_r - f_\delta\,,\ Au_{m+1,r} - f_\delta)}{\|Au_{m+1,r} - f_\delta\|} = \delta\,. \tag{21}$$

This rule was proposed and studied in [10] for case $m = 1$ (the ordinary Tikhonov method) and in [5] for general case $m \geq 1$ (see also [11]). In these works it was proved the monotonicity of function $d_{ME}(r)$ in (21), the property (18) and relation $r_{ME} \geq r_{RG}$, which leads to inequality $\|u_{r_{ME}} - u_*\| \leq \|u_{r_{RG}} - u_*\|$ (r_{RG} is parameter from (16); the last two inequalities are strong, if $A^*(f_\delta - Au_0) \neq 0$). Hence various error estimates in [9, 2–4] for $\|u_{r_{RG}} - u_*\|$ hold for $\|u_{r_{ME}} - u_*\|$ as well.

We note that basic idea of rules (14), (15) in [2] was to minimize the error bound for $\|u_r - u_*\|$, but ME-rule tries to minimize the norm of error $\|u_r - u_*\|$ itself by given noise level δ.

Note also that for method (6) the rules (12)–(14), (20) with $b = 1$, $\gamma = \overline{\gamma} = 1$ coincide.

c) ME-rule in iterative method (7). Consider iterative method (7). We start with simple lemma, using notation

$$r_n := Au_n - f_\delta\,.$$

Lemma 1. *If* (8) *holds, then*

$$\|r_{n+1}\|^2 \leq (r_n, r_{n+1}) \leq \|r_n\|^2 \qquad (\forall n \geq 0)\,. \tag{22}$$

Proof. We use the equality

$$r_{n+1} = (I - g(AA^*)AA^*)r_n \tag{23}$$

and notation $h(AA^*) := I - g(AA^*)AA^*$. From (8) follows $\|h(AA^*)\| \leq 1$, from (23) we have

$$\begin{aligned}\|r_{n+1}\|^2 &= \|h(AA^*)r_n\|^2 \leq \|h^{1/2}(AA^*)\|^2\|h^{1/2}(AA^*)r_n\|^2 \leq \|h^{1/2}(AA^*)r_n\|^2 \\ &= (r_n, r_{n+1}) \leq \|r_n\|\,\|r_{n+1}\| = \|r_n\|\,\|h(AA^*)r_n\| \leq \|r_n\|^2. \quad \square\end{aligned}$$

In the following we assume $r_1 \neq 0$. Then $r_0 \notin \mathcal{N}(h(AA^*)) = \mathcal{N}(h^n(AA^*))$ and (23) gives $r_n = h^n(AA^*)r_0 \neq 0$ $(\forall n \geq 0)$. Then the functional

$$\begin{aligned}d_{ME}(n) &\equiv \frac{(A(u_n + u_{n+1})/2 - f_\delta\,,\, g(AA^*)(Au_n - f_\delta))}{\|g(AA^*)(Au_n - f_\delta)\|} \\ &\equiv \frac{(r_n + r_{n+1}\,,\, g(AA^*)r_n)}{2\|g(AA^*)r_n\|}\end{aligned} \tag{24}$$

is well-defined. We formulate ME-rule.

ME-rule. Choose n_{ME} as the first index $n = 0, 1, \ldots,$ for which $d_{ME}(n) \leq \delta$.

The choice of n by this rule is possible, while due to (24), (22)

$$d_{ME}(n) \leq \|r_n + r_{n+1}\|/2 \leq \|r_n\| = \|Au_n - f_\delta\| \tag{25}$$

and $\lim_{n\to\infty} \|Au_n - f_\delta\| \leq \delta$ (see [12–15]).

Theorem 2. *Defined above ME-rule for method* (7) *satisfies condition* (19) *and we have*

$$\|u_{n_{ME}} - u_*\| < \|u_n - u_*\| \qquad (\forall n < n_{ME})\,. \tag{26}$$

Proof. Using equalities (23) and $g(A^*A)A^* = A^*g(AA^*)$ (see e.g. [12, 13, 15]), from (7) and ME-rule we have for $n = 0, 1, \ldots, n_{ME} - 1$

$$\begin{aligned}\|u_n - u_*\|^2 - \|u_{n+1} - u_*\|^2 &= (2(u_n - u_*) - g(A^*A)A^*r_n\,,\, g(A^*A)A^*r_n) \\ &= (2A(u_n - u_*) - Ag(A^*A)A^*r_n\,,\, g(AA^*)r_n) \\ &= (2(Au_n - f) - g(AA^*)AA^*r_n\,,\, g(AA^*)r_n) \\ &= (r_n + (I - g(AA^*)AA^*)r_n\,,\, g(AA^*)r_n) - 2(f - f_\delta\,,\, g(AA^*)r_n) \\ &\geq (r_n + r_{n+1}\,,\, g(AA^*)r_n) - 2\delta\|g(AA^*)r_n\| \\ &= 2(d_{ME}(n) - \delta)\|g(AA^*)r_n\| > 0\,.\end{aligned} \tag{27}$$

It proves (19), from (19) follows (26). □

Remark 1. Consider case $\delta = 0$. Then from (27) we have

$$\|u_n - u_*\|^2 - \|u_{n+1} - u_*\|^2 = (r_n + r_{n+1}\,,\, g(AA^*)r_n)$$
$$= \|[(2I - g(AA^*)AA^*)g(AA^*)]^{1/2} r_n\|^2 > 0,$$

thus condition (8) guarantee monotonical decrease of error for all $n \geq 0$. At that $\|u_n - u_*\| \to 0$ $(n \to \infty)$ (see e.g. [12–15]).

Remark 2. Note that $\|Au_n - f_\delta\| > \delta$ for $\forall n < \infty$ is possible only in case $\|f_\delta - f\| = \delta$, $Qf_\delta = f$, where $Q \in \mathcal{L}(F, F)$ is orthogonal projector onto $\mathcal{R}(A) \subseteq F$ (see [15], p. 78). In this case u_n in (7) coincides with u_n^0, get from (7), replacing f_δ there by f; hence $\|u_n - u_*\| = \|u_n^0 - u_*\| \to 0$ for $n \to \infty$ (as in noise-free case). Therefore in exceptional case, when ME-rule does not give final n_{ME}, one may use convention $n_{ME} = \infty$ with $u_\infty = u_*$.

3. Comparison of rules for a posteriori choice of n in iterative method (7)

In the following we compare for method (7) stopping indexes n_D, n_{EG} and n_{ME}, get by discrepancy principle (D-principle), by rule of Engl and Gfrerer (EG-rule) and by ME-rule respectively. Constants b and κ below are constants in D-principle (12) and EG-rule (15) respectively.

We give also some conditions, by which for considered rules the following Assertion A holds.

Assertion A. The approximations u_n converge: $\|u_n - u_*\| \to 0$ for $\delta \to 0$; if

$$u_* - u_0 = (A^*A)^{p/2}v\,, \qquad v \in H\,, \qquad \|v\| \leq E\,, \tag{28}$$

then the order optimal error estimate

$$\|u_n - u_*\| \leq \text{const}\, E^{1/(p+1)}\delta^{p/(p+1)}$$

holds.

Lemma 3. *1) If for some $k \geq 0$ the inequality*

$$2\|r_{n+k}\|\,\|g(AA^*)r_n\| \leq b(r_n + r_{n+1}\,,\, g(AA^*)r_n) \quad (\forall n \leq n_{ME}) \tag{29}$$

holds, then

$$d_D(n+k) \leq d_{ME}(n) \qquad (\forall n \leq n_{ME})\,, \qquad n_D - k \leq n_{ME}\,; \tag{30}$$

if (29) *holds for* $k = 0$, *then*

$$\|u_{n_{ME}} - u_*\| \leq \|u_{n_D} - u_*\|\,. \tag{31}$$

2) For $b = 1$ *in D-principle we have*

$$\begin{array}{llll} d_{EG}(n) & \leq d_D(n)\,, & d_{ME}(n) & \leq d_D(n) \quad (\forall n \geq 0)\,, \\ n_{EG} & \leq n_D\,, & n_{ME} & \leq n_D\,. \end{array} \tag{32}$$

3) If for some $k \geq 0$ *the inequality*

$$2\kappa\|r_{n+k}\|^2 \leq b^2(r_n + r_{n+1}\,,\, g(AA^*)r_n) \qquad (\forall n \leq n_{EG}) \tag{33}$$

holds, then

$$d_D(n+k) \leq d_{EG}(n) \qquad (\forall n \leq n_{EG})\,, \qquad n_D - k \leq n_{EG}\,. \tag{34}$$

4) If

$$S_n \equiv \frac{\kappa(r_n + r_{n+1}\,,\, g(AA^*)r_n)}{2\|g(AA^*)r_n\|^2} \geq 1 \qquad (\forall n \geq 0)\,, \tag{35}$$

then

$$d_{EG}(n) \leq d_{ME}(n) \qquad (\forall n \geq 0)\,, \qquad n_{EG} \leq n_{ME}\,, \tag{36}$$

$$\|u_{n_{ME}} - u_*\| \leq \|u_{n_{EG}} - u_*\|\,. \tag{37}$$

If in (35) $S_n \leq 1$ *(*$\forall n \geq 0$*), then*

$$d_{ME}(n) \leq d_{EG}(n) \qquad (\forall n \geq 0)\,, \quad n_{ME} \leq n_{EG}\,. \tag{38}$$

5) The assertion A holds for $n \in \{n_D, n_{EG}\}$*, but also for* $n = n_{ME}$*, if* (29) *holds for* $k = 0$ *and some* $b \geq 1$ *or if* (35) *holds for some* $\kappa \geq \overline{\kappa}$.

Proof. 1) Let (29) holds for some $k \geq 0$. Then from (12), (24) for $n = n_{ME}$ we have $d_D(n_{ME} + k) \leq d_{ME}(n_{ME})$. This inequality with $d_{ME}(n_{ME}) \leq \delta$ (see ME-rule) gives $d_D(n_{ME} + k) \leq \delta$, hence discrepancy principle gives $n_D \leq n_{ME} + k$. For $k = 0$ from $n_D \leq n_{ME}$ and (26) follows (31).
2) For $b = 1$ we have from (25) $d_{ME}(n) \leq d_D(n)$ and from (15), (22) $d_{EG}(n) \leq \|A(u_n + u_{n+1})/2 - f_\delta\| \leq d_D(n)$. Corresponding inequalities $n_{EG} \leq n_D$, $n_{ME} \leq n_D$ for indexes follow from parameter choice rules.

3), 4) Proofs of assertions 3), 4) are analogous to proofs of assertions 1), 2). Inequality (37) follows from inequalities $n_{EG} \leq n_{ME}$ (see (36)) and (26).
5) Assertion A for $n \in \{n_D, n_{EG}\}$ is well-known (see [12, 14, 2]). This gives assertion A for $n = n_{ME}$ due to (31), if (29) holds for $k = 0$ and some $b \geq 1$, and due to (37), if (35) holds for some $\kappa \geq \overline{\kappa}$. □

In the following Proposition 4 assumptions of assertions 3), 4) of Lemma 3 are given in terms of function $g(\lambda)$.

Proposition 4. *1) If the inequality*

$$g(\lambda) \leq 2\kappa/(2+\kappa\lambda) \qquad (\forall\lambda \in [0,a])\,, \tag{39}$$

holds, then (36), (37) *hold. If the inequality*

$$g(\lambda) \geq 2\kappa/(2+\kappa\lambda) \qquad (\forall\lambda \in [0,a])\,, \tag{40}$$

holds, then (38) *holds.*
2) If the inequality

$$(1-\lambda g(\lambda))^2 \leq [1+2\kappa\lambda/b^2]^{-1} \qquad (\forall\lambda \in [0,a])\,, \tag{41}$$

holds, then $d_D(n+1) \leq d_{EG}(n)$ ($\forall n \geq 0$), $n_D - 1 \leq n_{EG}$.
3) If (39), (41) *hold for some $\kappa \geq \overline{\kappa}$, then $d_D(n+1) \leq d_{ME}(n)$ ($\forall n \geq 0$), $n_D - 1 \leq n_{ME}$.*

Proof. 1) We introduce notations $h(\lambda) := 1 - \lambda g(\lambda)$ and $B := AA^*$. We have

$$\frac{d_{ME}(n)}{d_{EG}(n)} = \frac{\kappa^{1/2}||[(h^{2n}(B)+h^{2n+1}(B))g(B)/2]^{1/2}r_0||}{||h^n(B)g(B)r_0||}\,. \tag{42}$$

The quotient in last equality depends on function

$$G(\lambda) := \frac{[h^{2n}(\lambda)+h^{2n+1}(\lambda)]g(\lambda)}{2h^{2n}(\lambda)g^2(\lambda)} = \frac{1}{g(\lambda)} - \frac{\lambda}{2}\,.$$

If (39) holds, then $G(\lambda)\cdot\kappa \geq 1$ ($\forall\lambda \in [0,a]$), hence (42) gives (36), which due to (26) gives (37). If (40) holds, then $G(\lambda)\cdot\kappa \leq 1$ ($\forall\lambda \in [0,a]$), hence (42) gives (38).
2) Assertion $d_D(n+1) \leq d_{EG}(n)$ has form

$$\kappa||h^{n+1}(B)r_0||^2 \leq b^2||[(h^{2n}(B)+h^{2n+1}(B))g(B)/2]^{1/2}r_0||^2\,.$$

This inequality is true, if it holds

$$2\kappa h^{2n+2}(\lambda) \leq b^2 g(\lambda)[h^{2n}(\lambda)+h^{2n+1}(\lambda)] \qquad (\forall\lambda \in [0,a])\,.$$

The last inequality is equivalent to (41).
3) Assertion 3) is corollary of assertions 1), 2). □

4. Comparison of rules for choice of n in iteration schemes (9), (10).

In this section we apply Lemma 3 to iteration schemes (9), (10). In explicit iteration scheme (9) (called also as Richardson method or Landweber method (see [2, 3, 6]) $g(AA^*) = \mu$ =const, hence the functional $d_{ME}(n)$ in (24) has the form

$$d_{ME}(n) \equiv \frac{(A(u_n + u_{n+1})/2 - f_\delta\,,\, Au_n - f_\delta)}{\|Au_n - f_\delta\|}\,.$$

In implicit iteration scheme (10) (called also as Lardy's method (see [2])) $g(AA^*) = (\rho + AA^*)^{-1}$, hence

$$g(AA^*)(Au_n - f_\delta) = \rho^{-1}(Au_{n+1} - f_\delta) \tag{43}$$

and the functional $d_{ME}(n)$ in (24) has the form

$$d_{ME}(n) \equiv \frac{(A(u_n + u_{n+1})/2 - f_\delta\,,\, Au_{n+1} - f_\delta)}{\|Au_{n+1} - f_\delta\|}\,.$$

The choice of n by ME-rule do not need any constant, but remind that discrepancy principle (D-principle) uses constant $b \geq 1$, rule of Engl and Gfrerer (EG-rule) uses constant $\kappa \geq \overline{\kappa} \equiv \sup\{g(\lambda)\,|\,0 \leq \lambda \leq a\}$: in scheme (9) $\overline{\kappa} = \mu$, in scheme (10) $\overline{\kappa} = \rho^{-1}$.

Theorem 5. *For iteration schemes* (9), (10) *it holds:*

1) it holds $n_D - 1 \leq n_{ME}$ and

$$\|u_n - u_*\| < \|u_{n-1} - u_*\| \qquad (n = 1, 2, \ldots, n_D - 1)\,; \tag{44}$$

if D-principle uses $b \geq 2$ in scheme (9), *then the inequality in* (44) *holds for $n = n_D$ as well and it holds* (31);

2) if in D-principle $b = 1$ is used, then $n_{ME} \leq n_D$, $n_{EG} \leq n_D$;

3) if EG-rule and D-principle use κ and b with $\kappa \leq \overline{\kappa} b^2$, then $n_D - 1 \leq n_{EG}$;

4) the inequalities (36), (37) *hold in scheme* (10) *always, in scheme* (9) *in case $\kappa \geq 2\mu$; for $\kappa = \mu$ in scheme* (9) *the inequality* (38) *holds;*

5) assertion A holds for $n \in \{n_D, n_{EG}, n_{ME}\}$.

Proof. 1) Consider scheme (9). Due to inequalities (see (22))

$$2\|r_n\|\,\|r_{n+1}\| \le \|r_n\|^2 + \|r_{n+1}\|^2 \le \|r_n\|^2 + (r_n, r_{n+1}) \qquad (\forall n \ge 0) \qquad (45)$$

the inequality (29) holds for $g(\lambda) = \mu$, $k = 1$, $b \ge 1$. From (30) with $k = 1$ we get $n_D - 1 \le n_{ME}$; from (19) follows (44). In case $b \ge 2$ (29) holds for $k = 0$ as well, giving $n_D \le n_{ME}$ and (31), (44) for $n \le n_D$.

Consider scheme (10). The inequality (29) has in case $k = 1$ due to (43) the form $2\|r_{n+1}\|^2 \le b(r_n, r_{n+1}) + b\|r_{n+1}\|^2$, which holds (see (22)). From (30), (19) we get (44).

2) See assertion 2) of Lemma 3.

3) Due to (22), (43) the inequality (33) holds for $k = 1$, $\kappa \le b^2\overline{\kappa}$, giving (34) with $k = 1$.

4) Assertion 4) follows from assertion 4) of Lemma 3, while for $g(\lambda) = \mu$ we have (see (22))

$$\frac{\kappa}{2\mu} \le S_n \equiv \frac{\kappa[\|r_n\|^2 + (r_n, r_{n+1})]}{2\mu\|r_n\|^2} \le \frac{\kappa}{\mu}\,. \qquad (46)$$

and for $g(\lambda) = (\rho + \lambda)^{-1}$ we have

$$S_n \equiv \frac{\rho\kappa[(r_n, r_{n+1}) + \|r_{n+1}\|^2]}{(2\|r_{n+1}\|^2)} \ge 1\,.$$

5) Use assertion 5) of Lemma 3 with (46) and (45) in schemes (9), (10) respectively.

Remark 1. In addition to assertion 5) of Theorem 5 one may give another error estimate for iteration schemes (9), (10). Namely under some further assumptions (see [9, 2, 3]) for $n \in \{n_D, n_{EG}, n_{ME}\}$ the quasioptimality relation

$$\sup_{\|f_\delta - f\| \le \delta} \|u_n - u_*\| \le \text{const} \sup_{\|f_\delta - f\| \le \delta} \inf_{k \in \mathbb{N}} \|u_k - u_*\| \qquad (47)$$

holds. In [9, 2, 3] (47) is proved for $n \in \{n_D, n_{EG}\}$ (for $n = n_D$ with arbitrary $b > 1$ in D-principle, for $n = n_{EG}$ with arbitrary $\kappa > \overline{\kappa}$ in EG-rule), but it holds also for $n = n_{ME}$: in scheme (10) due to (37), in scheme (9) due to (31) for $b \ge 2$ (or due to (37) for $\kappa \ge 2\mu$).

Remark 2. Some assertions of Theorem 5 can be get from Proposition 4 as well. Namely for iteration scheme (10) inequality (39) holds always, inequality (41) holds for $\kappa\rho \le b^2$. For iteration scheme (9) inequalities (39) and (40) hold for $\kappa \ge 2\mu/(2 - \mu\|A\|^2)$ and $\kappa = \mu$ respectively, inequality (41) hold for $\kappa \le \mu b^2$, if $\mu \in (0, 3/2\|A\|^{-2})$.

5. Monotonicity of error and choice of constants b, γ, κ in rules (12)–(15)

a) The problem of choice of constants b, γ, κ in rules (12)–(15) is important. In discrepancy principle for non-iterative mehods r is choosen from equation $\|Au_r - f_\delta\|/b = \delta$. The recommendation to take here $b > 1$ (or $b \geq 1$) is close to recommendation to take here parameter r such that $\|Au_r - f_\delta\| > \delta$ (or $\|Au_r - f_\delta\| \geq \delta$). It leaves possibility to choose too different b (e.g. also $b = 1000$ is not excluded) and r. Reasonable choice of b needs specification, in what case choice $b = 1$ is good and what is reasonable upper bound for b. Somewhat surprisingly in investigations of ill-posed problems for choice of b only few attention has been given. Note that discrepancy principle with $b = 1$ is investigated in [12, 14, 15].

In m-iterated Tikhonov method (5) we have for all $b \geq 1$ in (13) (i.e. for all $\gamma \geq \overline{\gamma} = m$ in (14)) $r_{RG} \leq r_{ME}$ ($r_{RG} < r_{ME}$, if $A^*(f_\delta - au_0) \neq 0$).Due to strong monotone decrease of error $\|u_r - u_*\|$ for $r \in (0, r_{ME})$ (see (18)) the best choice of b here is $b = 1$ (the best choice of γ in (14) is $\gamma = m$). Increase of b leads to decrease of (monotone) function $d_{RG}(r)$ in (16), which leads to decrease of r_{RG} and this leads to increase of error $\|u_{r_{RG}} - u_*\|$.

In the following we show, that situation in iteration methods (9), (10) is analogous.

b) On earlier results about monotonicity of error in scheme (7). Let us discuss the result (44). For implicit iteration scheme (10) and other methods (7) except (9) we do not know earlier results about bounds n_* in monotonicity property

$$\|u_n - u_*\| < \|u_{n-1} - u_*\| \quad \text{for} \quad n \leq n_* \tag{48}$$

by a posteriori choice of n. For explicit iteration scheme (9) it was shown in [1] (see also [3], p. 157), that (48) holds with $n_* = n_D$, where n_D is index coming from the discrepancy principle with $b = 2$. Result (44) with $b = 1$ essentially increases this bound n_*. The monotonicity property (48) plays an important role also by extension of results of iteration scheme (9) (the Landweber method) about linear problems to nonlinear problems. The stopping index n in Landweber method for nonlinear problems is choosed by discrepancy principle with $b \geq 2$ by reason (see [3], p. 280–284), that in linear problems the property (48) was known only for $n_* = n_D$ with $b \geq 2$ in discrepancy principle.

c) Smaller $b > 1$, $\kappa > \overline{\kappa}$ lead to smaller error in (9), (10). Denote by n_D^*, n_{EG}^* indexes, get by discrepancy principle with $b = 1$ and by EG-rule with $\kappa = \overline{\kappa}$ respectively. Theorem 5 shows that if in discrepancy principle and EG-rule minimal constants $b = 1$, $\kappa = \mu$ in scheme (9), $\kappa = \rho^{-1}$ in scheme (10) are used, all considered three rules give almost equivalent result:

$$n_D^* - 1 \leq n_{ME} \leq n_D^*, \quad n_D^* - 1 \leq n_{EG}^* \leq n_D^*.$$

The increase of constants b and κ leads to decrease of corresponding indexes n_D, n_{EG}. Taking into account, that error $\|u_n - u_*\|$ is monotonically decreasing for $n \leq n_{ME}$ (see (26)), also for $n \leq n_D^* - 1$ (see (44)), the decrease of index below n_{ME} (below $n_D^* - 1$) leads to increase of error. Particulary, in explicit iteration scheme (9) constants $b > 2$ and $\kappa > 2\mu$ in discrepancy principle and in rule of Engl and Gfrerer respectively are too large (see Theorem 5.1), 4)).

In implicit iteration scheme (10) we have according to Theorem 5.4)

$$n_{EG} \leq n_{EG}^* \leq n_{ME}, \quad \|u_{n_{ME}} - u_*\| \leq \|u_{n_{EG}}^* - u_*\| \leq \|u_{n_{EG}} - u_*\|,$$

hence the best choice of constant κ in EG-rule is $\kappa = \overline{\kappa} \equiv \rho^{-1}$.

Note that in [6] numerical results were reported by solving the test problem by various iterative methods (methods of Landweber and Chebyshev, Brakhage's ν-method), and for finding the stopping index by discrepancy principle the constant $b = 1$ "turned out to be the most satisfactory choice for all methods" ([6], p. 370).

d) case $b = 1$ in discrepancy principle for (7). Let us consider for iteration scheme (7) discrepancy principle with $b = 1$. In [12, 13, 15] the assertion A was proved by restrictions (17) for function $g(\lambda)$. For scheme (10) conditions (17) is fulfilled. For explicit iteration scheme (9) condition (17) is fulfilled for $\mu \in (0, \|A\|^{-2}]$ and violated for $\mu \in (\|A\|^{-2}, 2\|A\|^{-2})$; validity of assertion A for $\mu \in (\|A\|^{-2}, 2\|A\|^{-2})$ is formulated in [12, p.98] as open problem. We try to solve it.

Proposition 6. *Let* (29) *holds for* $b = 1$ *and some* $k \geq 0$. *Then discrepancy principle with* $b = 1$ *gives* $n = n_D^*$, *for which assertion* A *holds.*

Proof. Consider case $n_D^* < \infty$. From (29) with $b = 1$ we have (30): $n_D^* \leq n_{ME} + k$. From (7) and monotonical decrease of discrepancy $\|Au_n - f_\delta\|$ we have for $n \leq n_D^*$

$$\begin{aligned} \|u_{n_D^*} - u_*\| &= \Big\|u_n - u_* - g(A^*A)A^* \sum_{i=n}^{n_D^*-1} (Au_i - f_\delta)\Big\| \\ &\leq \|u_n - u_*\| + (n_D^* - n)\|g(A^*A)A^*\| \, \|Au_n - f_\delta\|. \end{aligned} \tag{49}$$

Denote $\kappa' = \sup\{\lambda^{1/2} g(\lambda) \,|\, 0 \le \lambda \le a\}$. Fix some $b > 1$. The discrepancy principle with this b gives $n_D \le n_D^*$. If $n_D > n_{ME}$, the inequality (49) with $n = n_D$ and $||Au_{n_D} - f_\delta|| \le b\delta$ gives

$$||u_{n_D^*} - u_*|| \le ||u_{n_D} - u_*|| + k\kappa' b\delta\,. \tag{50}$$

If $n_D \le n_{ME}$, the inequality (49) with $n = n_{ME}$ and inequalities

$$||u_{ME} - u_*|| \le ||u_{n_D} - u_*||\,, \quad ||Au_{n_{ME}} - f_\delta|| \le ||Au_{n_D} - f_\delta|| \le b\delta$$

gives (50) as well. The assertion A holds for $n = n_D$ (see [12, 13, 15]), due to (50) it holds for $n = n_D^*$ as well.

In case $n_D^* = \infty$ (i.e. $||Au_n - f_\delta|| > \delta$ for $\forall n < \infty$) we use Remarks 1, 2 of Section 2. Situation $n_D^* = \infty$ is possible only in case $Qf_\delta = f$, and then $||u_n - u_*|| \to 0$ monotonically for $n \to \infty$ as in noise-free case, from $n_D^* \ge n_D$ follows $||u_{n_D^*} - u_*|| < ||u_{n_D} - u_*||$. □

Corollary. If in explicit iteration scheme (9) with $\mu \in (0, 2||A||^{-2})$ index n is choosed by discrepancy principle with $b = 1$, the assertion A holds (inequality (29) for $g(\lambda) = \mu$, $k = 1$ is stated in (45)).

e) Conclusion: how to choose n in iteration method (7). For method (7), satisfying (29) with $b = 1$, on the base at Proposition 6 one may recommend to use the discrepancy principle with $b = 1$. After finding n_D^* one may in (9), (10) additionally choose between n_D^* and $n_D^* - 1$: choose $n = n_D^*$ in case $d_{ME}(n_D^*) \le \delta$, choose $n = n_D^* - 1$ otherwise.

But note that if (28) holds, in case $b > 1$ the bound $n_D \le \text{const}\,[(b - 1)\delta/E]^{-2/(p+1)}$ is valid (see [12–15]) but there is no bound for n_D^*. To decrease stopping index (i.e. to decrease computational costs and influence of computational errors), one may stop iterations in (7) also by some $n < n_{ME}$ or $n < n_D^*$ (e.g. take $n = n_D$ with little $b > 1$, e.g. with $b = 1.001$; or take stopping index n as the first $n = 0, 1, \ldots$ with $||Au_n - f_\delta|| \le \delta + \epsilon$ or $d_{ME}(n) \le \delta + \epsilon$, where ϵ is accuracy (tolerance) for computing $||Au_n - f_\delta||$ or $d_{ME}(n)$ respectively).

On the other hand, it is reasonable not to stop iterations earlier than guaranteed computable increase of accuracy on the right hand side of inequality (see (27))

$$||u_n - u_*||^2 - ||u_{n+1} - u_*||^2 \ge 2(d_{ME}(n) - \delta)||g(AA^*)(Au_n - f_\delta)||$$

of interest is (by given computational costs of one iteration).

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