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ON THE PARAMETER CHOICE IN THE REGULARIZED RITZ-GALERKIN METHOD

(Presented by G. Vainikko)

1. Introduction. Let H be a Hilbert space. Consider the equation

$$Au=f, \quad f\in \mathfrak{R}(A), \tag{1}$$

where $A\in \mathfrak{L}(H,H)$, $A=A^*\geqslant 0$ and only an approximation $f_\delta\in H$ with $\|f_\delta-f\|\leqslant\delta$ is available. We discretize the problem (1) by the Ritz-Galerkin method $A_hu_h=P_hf_\delta$, $A_h=P_hAP_h$, $u_h\in\mathfrak{R}(P_h)$, where $h>0$ and P_h is an orthogonal projection in H. If $\mathfrak{R}(A)$ is non-closed, the last equation needs regularization. Using standard methods for it with initial approximation $u_0\in H$ and regularization parameter r, we get approximation $u_{h,r}\in\mathfrak{R}(P_h)$. Let u_* be the solution of (1), closest to u_0. In this paper we specify convergence conditions and convergence rate for $u_{h,r}\to u_*$ ($\delta\to 0$, $h\to 0$), given in [1]. We show for the Lavrentiev method and its modifications that $\sum_{i=1}^{n} d_i u_{h,r_i}$ with proper d_i, $r_i\in\mathbf{R}$, $i=1,\ldots,n$ converges to smooth u_* faster than $u_{h,r}$. Applications to integral equations of the first kind are given.

Note that the regularized Ritz-Galerkin method was investigated also in [2–7], regularized projection methods for (1) with non-selfadjoint operators in [3, 7–14].

2. Standard regularization methods. Let $g_r:[0,a]\to\mathbf{R}$ ($r\geqslant 0$, $\|A\|\leqslant a$) be a Borel measurable function, satisfying the conditions

$$\sup_{0\leqslant\lambda\leqslant a}|g_r(\lambda)|\leqslant\gamma r, \quad r\geqslant 0, \tag{2}$$

$$\sup_{0\leqslant\lambda\leqslant a}\lambda^p|1-\lambda g_r(\lambda)|\leqslant\gamma_p r^{-p}, \quad r>0, \quad 0\leqslant p\leqslant p_0, \quad p_0\geqslant 0, \tag{3}$$

where p_0, γ and γ_p are positive constants. Define

$$u_r=(I-Ag_r(A))u_0+g_r(A)f_\delta. \tag{4}$$

A proper choice of $g_r(\lambda)$ in (4) gives the following standard methods.

1) The method of iteration (explicit scheme). Let $\mu\in(0,1/a)$,

$$u_r=(I-\mu A)u_{r-1}+\mu f_\delta, \quad r=1,2,\ldots.$$

Here $g_r(\lambda)=\lambda^{-1}[1-(1-\mu\lambda)^r]$, $p_0=\infty$.

2) The method of iteration (implicit scheme). Let $\mu>0$ and

$$u_r=(A+\mu I)^{-1}(\mu u_{r-1}+f_\delta), \quad r=1,2,\ldots.$$

Here $g_r(\lambda)=\lambda^{-1}[1-(\mu/(\mu+\lambda))^r]$, $p_0=\infty$.

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3) The method of Lavrentiev

$$u_r = (r^{-1}I + A)^{-1} f_\delta. \tag{5}$$

Here $u_0 = 0$, $g_r(\lambda) = (\lambda + r^{-1})^{-1}$, $p_0 = 1$.

4) The iterated method of Lavrentiev. A natural number $m \geqslant 1$ is given and $u_{r,0} = u_0$. We find iteratively

$$u_{r,n} = (r^{-1}I + A)^{-1}(r^{-1}u_{r,n-1} + f_\delta), \quad n = 1, \dots, m; \quad u_r = u_{r,m}. \tag{6}$$

Here $g_r(\lambda) = \lambda^{-1}[1 - (1 + r\lambda)^{-m}]$, $p_0 = m$.

5) The generalized method of Lavrentiev. Let $q \geqslant 1$ and

$$u_r = (r^{-q}I + A^q)^{-1} A^{q-1} f_\delta. \tag{7}$$

Here $u_0 = 0$, $g_r(\lambda) = \lambda^{q-1}/(\lambda^q + r^{-q})$, $p_0 = q$.
The greatest value of p_0, for which inequality (3) holds, is named a qualification of method (4). It is well known (see e. g. [15]) that if

$$u_0 - u_* = A^p z, \quad \|z\| \leqslant \varrho, \tag{8}$$

then a proper choice of r gives for $p \leqslant p_0$

$$\|u_r - u_*\| \leqslant c(\varrho \delta^p)^{1/(p+1)}. \tag{9}$$

3. Extrapolation of regularization methods. Consider the following possibility for lifting the qualification of methods with $p_0 < \infty$.

Let Θ_i, $d_i \in \mathbb{R}$, $i = 1, \dots, n$, $n \geqslant 2$ satisfy $0 < \Theta_1 < \Theta_2 < \dots < \Theta_n = 1$, $\sum_{i=1}^{n} d_i = 1$. Let the function $g_r(\lambda)$ satisfy (2), (3) with $p_0 < \infty$ and u_r be corresponding element (4). Then the function $g_{1,r}(\lambda) = \sum_{i=1}^{n} d_i g_{r_i}(\lambda)$ with $r_i = \Theta_i r$ satisfies (2), (3) also with certain p_1 instead of p_0, where $p_1 \geqslant p_0$ due to $1 - \lambda g_{1,r}(\lambda) = \sum_{i=1}^{n} d_i (1 - \lambda g_{r_i}(\lambda))$. The corresponding analogue of (4) has the form

$$v_r := (I - A g_{1,r}(A)) u_0 + g_{1,r}(A) f_\delta = \sum_{i=1}^{n} d_i (I - A g_{r_i}(A)) u_0 + \sum_{i=1}^{n} d_i g_{r_i}(A) f_\delta,$$

$$\Rightarrow v_r = \sum_{i=1}^{n} d_i u_{r_i}. \tag{10}$$

A proper choice of $d_i = d_i(\Theta_1, \dots, \Theta_{n-1})$, $i = 1, \dots, n$ may guarantee that the qualification p_1 of approximation (10) is higher than p_0 of the initial method. If (8) holds, then we have for $u_r = v_r$ estimate (9) with $p \leqslant p_1$ choosing $r = \mathrm{const} \cdot \delta^{-1/(p+1)}$, and with $p \leqslant p_1 - 1$ (provided $p_1 > 1$) choosing r by the discrepancy principle $\|Av_r - f_\delta\| = b\delta$, $b = \mathrm{const} > 1$.

Let us consider algorithms based on methods 3), 4), 5). Then we have $1 - \lambda g_r(\lambda) = [1 + (r\lambda)^q]^{-m}$ with $q = m = 1$, $q = 1$ and $m = 1$, respectively. For $\varepsilon := (r\lambda)^{-q} < 1$ we get the Taylor series

$$1 - \lambda g_r(\lambda) = (1 + \varepsilon^{-1})^{-m} = \sum_{k=m}^{\infty} c_k \varepsilon^k,$$

$$c_k = (-1)^{k-m}(k-1)!/[(m-1)!(k-m)!],$$

$$1 - \lambda g_{1,r}(\lambda) = \sum_{i=1}^{n} d_i \sum_{k=m}^{\infty} c_k \Theta_i^{-kq} \varepsilon^k = \sum_{k=m}^{\infty} \Big[\sum_{i=1}^{n} d_i \Theta_i^{-kq} \Big] c_k \varepsilon^k$$

Let $d_1, \dots, d_n$ be the solution of the system of linear equations $\sum_{i=1}^{n} d_i = 1$, $\sum_{i=1}^{n} d_i \Theta_i^{-kq} = 0$, $k = m, \dots, m+n-2$. Then $1 - \lambda g_{1,r}(\lambda) = O(\varepsilon^{m+n-1}) = O((r\lambda)^{-q(m+n-1)})$, hence $g_{1,r}(\lambda)$ satisfies (3) with $p_1 = q(m+n-1)$. Using the first row expansion of the determinant of the system and the formula $V(x_1, \dots, x_n) = \prod_{k>l} (x_k - x_l)$ for the Vandermonde determinant V, we have

$$d_i = D_i / \sum_{j=1}^{n} D_j,$$

$$D_i = (-1)^{i-1} \prod_{j=1,\, j \neq i}^{n} \Theta_j^{-mq} V(\Theta_1^{-q}, \dots, \Theta_{i-1}^{-q}, \Theta_{i+1}^{-q}, \dots, \Theta_n^{-q}) =$$

$$= (-1)^{i-1} \prod_{j=1,\, j \neq i}^{n} \Theta_j^{-mq} \prod_{k>l,\, k \neq i,\, l \neq i} (\Theta_k^{-q} - \Theta_l^{-q}), \quad i = 1, \dots, n. \tag{11}$$

If $m = 1$, then it holds also that

$$d_i = D_i / V(\Theta_1^{-q}, \dots, \Theta_n^{-q}) = D_i / \prod_{k>l} (\Theta_k^{-q} - \Theta_l^{-q}) =$$

$$= \prod_{j=1,\, j \neq i}^{n} [1 - (\Theta_j/\Theta_i)^q]^{-1}, \quad i = 1, \dots, n.$$

Another approach for $m = 1$ also gives coefficient $\gamma_{p_1} = \prod_{i=1}^{n} \Theta_i^{-q}$ in (3). Namely, interpolating the function $(1 + \varepsilon)^{-1}$ by the Lagrange polynomial

$$\sum_{i=1}^{n} d_i(\varepsilon)(1 + \varepsilon_i)^{-1}, \quad d_i(\varepsilon) = \prod_{j=1,\, j \neq i}^{n} [(\varepsilon - \varepsilon_j)/(\varepsilon_i - \varepsilon_j)], \quad i = 1, \dots, n,$$

for $\varepsilon_i = (r_i \lambda)^{-q}$ and $\varepsilon = 0$ we have

$$d_i(0) = d_i, \quad (1 + \varepsilon_i)^{-1} = \lambda^q / (\lambda^q + r_i^{-q}) = \lambda g_{r_i}(\lambda),$$

$$|1 - \lambda g_{1,r}(\lambda)| = \Big|1 - \sum_{i=1}^{n} d_i (1 + \varepsilon_i)^{-1}\Big| \leqslant \prod_{i=1}^{n} \varepsilon_i = (r\lambda)^{-qn} \prod_{i=1}^{n} \Theta_i^{-q}.$$

6) The (iterated and) extrapolated method of Lavrentiev. Find u_{r_i} by (5) or (6) and v_r by (10) with d_i, $i = 1, \dots, n$ from (11) with $q = 1$. Here $g_{1,r}(\lambda) = \lambda^{-1}\{1 - \sum_{i=1}^{n} d_i (1 + \Theta_i \lambda r)^{-m}\}$, $p_1 = m + n - 1$.

7) The extrapolated (generalized) method of Lavrentiev. Find u_{r_i} by (5) or (7) and v_r by (10) with $d_i = \prod_{j=1,\, j \neq i}^{n} [1 - (\Theta_j/\Theta_i)^q]^{-1}$, $i = 1, \dots, n$. Here $g_{1,r}(\lambda) = \lambda^{-1}\{1 - \prod_{i=1}^{n} [1 + (\Theta_i \lambda r)^q]^{-1}\}$, $p_1 = qn$.

8) The extrapolated and iterated (generalized) method of Lavrentiev. Find v_r in method 7), put $u_{r,0} = v_r$, find by iterations (6) $u_r = u_{r,m}$. Here $g_{1,r}(\lambda) = \lambda^{-1}\{1 - \prod_{i=1}^{n} [1 + (\Theta_i \lambda r)^q]^{-m}\}$, $p_1 = qnm$.

In the case $n = 2$ methods 6), 7) have the form $v_r = (u_r - \Theta^s u_{\Theta r}) / (1 - \Theta^s)$ with $s = m$ and $s = q$, respectively; for $q = m = 1$, $\Theta = 1/2$ we get $v_r = 2u_r - u_{r/2}$ with u_r from (5). The last method was proposed in [5].

Note that for the non-selfadjoint problem (1) we get the analogues of methods 1)—8), replacing A and f_δ by A^*A and A^*f_δ, respectively. The analogue of method 7) with $q=1$ is the extrapolated method of Tikhonov, proposed for systems of linear equations in [16] (see also [17]) and for operator equations in [18, 19].

The considered ideas are particularly useful for the a-posteriori choice of the parameter, where u_{r_i}, $i=1, 2, \ldots$ are found, until for r_i a certain condition is fulfilled. The choice of $r=r_i$ by the usual discrepancy principle $\|Au_{r_i}-f_\delta\|=b\delta$, $b>1$ in the Lavrentiev method (5) leads to divergence, but the selection rule

$$\|Av_{r_i}-f_\delta\|=b\delta, \quad v_{r_i}:=(u_{r_i}-\Theta u_{r_{i-1}})/(1-\Theta), \quad \Theta=r_{i-1}/r_i, \quad b>1 \tag{12}$$

guarantees the convergence and in case (8) estimate (9) for $p\leqslant 1$. In method 4) rule (12) with Θ^m instead of Θ and the discrepancy principle give in case (8) estimate (9) for $p\leqslant m$ and $p\leqslant m-1$, respectively. It follows from Corollary of Theorem 5 in Section 7 (take $h=0$). Note that if problem (1) is non-selfadjoint and $u_*\in\mathfrak{R}(A^*A)$, then the Tikhonov method $u_r=(r^{-1}I+A^*A)^{-1}A^*f_\delta$ has order $O(\delta^{2/3})$ provided that r_i is chosen by the rule $(Au_{r_i}-f_\delta, Av_{r_i}-f_\delta)^{1/2}=b\delta$, $b>1$ with v_{r_i} from (12) (and order $O(\delta^{1/2})$, if the discrepancy principle is used).

A more detailed treatment of the extrapolation of regularization methods (especially with respect to a-posteriori parameter choice) can be found in a forthcoming paper of the author. About other rules for a-posteriori choice of r see [15, 20–22].

4. Regularization of the Ritz-Galerkin method. Let equation (1) be at first discretized by the Ritz-Galerkin method and after that regularized by the methods in Section 2. Then we get an analogue to (4)

$$u_{h,r}=(I-A_hg_r(A_h))P_hu_0+g_r(A_h)P_hf_\delta.$$

Using the same function $g_r(\lambda)$ as in methods 1)—5), we get their discrete analogues

1′) $u_{h,r}=(I-\mu A_h)u_{h,r-1}+\mu P_hf_\delta, \quad r=1, 2, \ldots,$

2′) $u_{h,r}=(A_h+\mu I)^{-1}(\mu u_{h,r-1}+P_hf_\delta), \quad r=1, 2, \ldots,$

3′) $u_{h,r}=(r^{-1}I+A_h)^{-1}P_hf_\delta,$

4′) $u_{h,r,0}=P_hu_0, \quad u_{h,r,n}=(r^{-1}I+A_h)^{-1}(r^{-1}u_{h,r,n-1}+P_hf_\delta), \quad n=1, \ldots, m,$
$u_{h,r}=u_{h,r,m},$

5′) $u_{h,r}=(r^{-q}I+A_h^q)^{-1}A_h^{q-1}P_hf_\delta.$

In the same way as methods 6)—8) form approximation (10) with u_{r_i} from methods 3)—5), their analogues 6′)—8′) use the approximation $v_{h,r}=\sum_{i=1}^{n}d_iu_{h,r_i}$ with u_{h,r_i} from 3′)—5′). Note that computational schemes for methods 1′)—5′) and their realization for integral equations of the first kind are given in [4].

5. Some auxiliary results. Denote

$$\xi_{h,p}:=\|(I-P_h)A^p\| \quad (p>0), \qquad \chi_{h,\varepsilon}:=\inf_{\alpha>0}(\xi_{h,\alpha}(1+\varepsilon^2))^{1/\alpha} \quad (\varepsilon>0), \tag{13}$$

$$S_{h,r}:=I-A_hg_r(A_h), \qquad T_{h,r}:=I-A_hg_{2,r}(A_h),$$

where $g_{2,r}:[0,a]\to\mathbf{R}$ $(r\geqslant 0)$ is a Borel measurable function.

From the inequality of moments (see [23])

$$\|D_1^\alpha z\|\leqslant\|D_1z\|^\alpha\|z\|^{1-\alpha} \quad (D_1\in\mathfrak{L}(H,H), \quad D_1=D_1^*\geqslant 0, \quad z\in H, \quad 0\leqslant\alpha\leqslant 1)$$

with $v=D_2w$, $D_2\in\mathfrak{L}(H,H)$, $w\in H$ it follows that

$$\|D_1^\alpha D_2\|\leqslant\|D_1D_2\|^\alpha\|D_2\|^{1-\alpha}. \tag{14}$$

Taking here $D_1=A^p$, $D_2=I-P_h$, $\alpha=q/p$, we get

$$\xi_{h,q}\leqslant\xi_{h,p}^{q/p} \quad (\forall q\leqslant p). \tag{15}$$

Lemma 1. *Let $k\in\mathbf{R}$ and function $G_{r,k}:[0,a]\to\mathbf{R}$, $r\geqslant 0$ satisfy for every $\lambda\in[0,a]$ the conditions*

$$\lambda|G_{r,k}(\lambda)|\leqslant\varkappa_1r^k, \tag{16}$$

$$\lambda^{1/2}|G_{r,k}(\lambda)|\leqslant\varkappa_*r^{k+1/2}, \tag{17}$$

where $\varkappa_1, \varkappa_*$ are positive constants. Then for every $\alpha\in(0,1/2]$, $\varepsilon\in(0,1/2]$

$$\|G_{r,k}(P_hAP_h)P_hA(I-P_h)\|\leqslant\max(\varkappa_1,\varkappa_*)r^k[\varepsilon^{-1}\min(1, r^\alpha\xi_{h,\alpha})+r^{1/2}\chi_{h,\varepsilon}^{1/2}]. \tag{18}$$

Proof. Denote $D_{r,k}:=P_hG_{r,k}(P_hAP_h)$, $x:=\|(I-P_h)AD_{r,k}\|$. Then

$$x\leqslant\|(I-P_h)A^\alpha\|\,\|A^{1-\alpha}D_{r,k}\|=\xi_{h,\alpha}\|A^{1-\alpha}D_{r,k}\| \quad (\forall\alpha\in(0,1/2]). \tag{19}$$

From (16), (17) it follows that

$$\|AD_{r,k}\|^2=\|P_hAD_{r,k}\|^2+\|(I-P_h)AD_{r,k}\|^2\leqslant(\varkappa_1r^k)^2+x^2,$$
$$\|A^{1/2}D_{r,k}\|^2=\|(A^{1/2}D_{r,k})^*A^{1/2}D_{r,k}\|=\|D_{r,k}A_hD_{r,k}\|\leqslant$$
$$\leqslant\|A_h^{1/2}D_{r,k}\|^2\leqslant(\varkappa_*r^{k+1/2})^2.$$

Using (14) with $D_1=A^{1/2}$, $D_2=A^{1/2}D_{r,k}$, $1-2\alpha$ instead of α and the last inequalities, we have

$$\|A^{1-\alpha}D_{r,k}\|\leqslant\|AD_{r,k}\|^{1-2\alpha}\|A^{1/2}D_{r,k}\|^{2\alpha}\leqslant[(\varkappa_1r^k)^2+x^2]^{(1-2\alpha)/2}(\varkappa_*r^{k+1/2})^{2\alpha}.$$

Combined with (19) it leads for $y:=(\varkappa_1r^k)^{-1}x$ to the relation

$$y\leqslant dr^\alpha\xi_{h,\alpha}(1+y^2)^{(1-2\alpha)/2}, \qquad d=(\varkappa_*/\varkappa_1)^{2\alpha}. \tag{20}$$

If $y\leqslant(2\varepsilon^2+\varepsilon^4)^{-1/2}$, then $y\leqslant\varepsilon^{-1}$ and (20) gives also

$$y\leqslant dr^\alpha\xi_{h,\alpha}(1+(2\varepsilon^2+\varepsilon^4)^{-1})^{1/2}\leqslant\varepsilon^{-1}dr^\alpha\xi_{h,\alpha} \quad (\forall\alpha\in(0,1/2], \quad \forall\varepsilon\in(0,1/2]).$$

If $y\geqslant(2\varepsilon^2+\varepsilon^4)^{-1/2}$, then $1+y^2\leqslant(1+\varepsilon^2)^2y^2$ and (20) gives

$$y\leqslant dr^\alpha\xi_{h,\alpha}(1+\varepsilon^2)^{1-2\alpha}y^{1-2\alpha}\Rightarrow y\leqslant d^{1/(2\alpha)}r^{1/2}\xi_{h,\alpha}^{1/(2\alpha)}(1+\varepsilon^2)^{(1-2\alpha)/(2\alpha)}.$$

It remains to take in the last inequality infimum over $\alpha\in(0,1/2]$ and to notice that if in (13) infimum is obtained by $\alpha=\alpha_*>1/2$, then $\xi_{h,1/2}\leqslant\xi_{h,\alpha_*}^{1/(2\alpha_*)}\leqslant\chi_{h,\varepsilon}^{1/2}$ due to (15).

Corollary. *For every $\varepsilon\in(0,1/2]$ it holds that*

$$\|g_r(A_h)P_hA(I-P_h)\|\leqslant c_1(\varepsilon^{-1}+r^{1/2}\chi_{h,\varepsilon}^{1/2}), \qquad c_1=\max(\gamma_0+1, [\gamma(\gamma_0+1)]^{1/2}). \tag{21}$$

If function $g_{2,r}(\lambda)$ satisfies

$$\sup_{0\leqslant\lambda\leqslant a}\lambda^p|1-\lambda g_{2,r}(\lambda)|\leqslant\gamma_{2,p}r^{-p}, \quad r>0, \quad 0\leqslant p\leqslant p_2, \quad p_2\geqslant 1, \quad \gamma_{2,p}=\mathrm{const}>0, \tag{22}$$

then it holds for every $\varepsilon\in(0,1/2]$, $\alpha\in(0,1/2]$ also that

$$\|T_{h,r}P_hA(I-P_h)\|\leqslant c_2(\varepsilon^{-1}r^{\alpha-1}\xi_{h,\alpha}+r^{-1/2}\chi_{h,\varepsilon}^{1/2}), \qquad c_2=\max(\gamma_{2,1/2},\gamma_{2,1}). \tag{23}$$

Proof. We get these estimates from (18), so as $G_{r,0}(\lambda)=g_r(\lambda)$ and $G_{r,-1}(\lambda)=1-\lambda g_{2,r}(\lambda)$ satisfy (16), (17) on the basis of (2), (3), (22).

In the following, c is a general positive constant, not depending on r, h, δ and ϱ, and taking different values in different inequalities.

Lemma 2. [1] *Let* $0\leqslant t\leqslant p_0$. *Then*

$$r^t\|A_h^t S_{h,r}P_hA^p\|\leqslant c(r^{-p'}+\xi_{h,p}),\quad p'=\min\{p,\ p_0-t\}. \tag{24}$$

6. A-priori choice of r. Theorem 3. *Let* $\varepsilon\in(0,1/2]$. 1. *Suppose that* $P_h\to I$ *pointwise,* $\xi_{h,1}\to 0$ $(h\to 0)$. *If* $r=r(\delta,h)$ *is chosen so that*

$$r\delta\to 0,\quad r\to\infty,\quad r\chi_{h,\varepsilon}\leqslant c,\quad (\delta\to 0,\ h\to 0), \tag{25}$$

then $u_{h,r}\to u_*$ $(\delta\to 0,\ h\to 0)$.

2. a) *If*

$$u_*=A^pz,\ \|z\|\leqslant\varrho\qquad u_0-u_*=A^pz_1,\ \|z_1\|\leqslant\varrho,\ p>0 \tag{26}$$

holds with $p\leqslant p_0$ *and* r *is chosen by the rule*

$$c[(\delta/\varrho)^{1/(p+1)}+\chi_{h,1/2}]\leqslant r^{-1}\leqslant c'[(\delta/\varrho)^{1/(p+1)}+\xi_{h,p}^{1/p}],$$

then

$$\|u_{h,r}-u_*\|\leqslant c[(\varrho\delta^p)^{1/(p+1)}+\varrho\xi_{h,p}]. \tag{27}$$

b) *If* (26) *holds with* $p>p_0$, *then the choice of* r *by the rule*

$$c[(\delta/\varrho)^{1/(p_0+1)}+(\chi_{h,\varepsilon}^{1/2}\xi_{h,p})^{2/(2p_0+1)}]\leqslant r^{-1}\leqslant$$
$$\leqslant c'[(\delta/\varrho)^{1/(p_0+1)}+(\chi_{h,\varepsilon}^{1/2}\xi_{h,p})^{2/(2p_0+1)}+\xi_{h,p}^{1/p_0}]$$

gives

$$\|u_{h,r}-u_*\|\leqslant c[(\varrho\delta^{p_0})^{1/(p_0+1)}+\varrho(\chi_{h,\varepsilon}^{1/2}\xi_{h,p})^{2p_0/(2p_0+1)}+\varepsilon^{-1}\varrho\xi_{h,p}], \tag{28}$$

for $\varepsilon=1/2$ *also*

$$\|u_{h,r}-u_*\|\leqslant c[(\varrho\delta^{p_0})^{1/(p_0+1)}+\varrho(\xi_{h,p}^{2p_0}\xi_{h,p_0})^{1/(2p_0+1)}. \tag{29}$$

Proof. We have

$$u_{h,r}-u_*=S_{h,r}P_h(u_0-u_*)-[I-g_r(A_h)P_hA(I-P_h)](I-P_h)u_*+$$
$$+g_r(A_h)P_h(f_\delta-f)$$

and estimating by (2), (21), the last inequality yields for every $\varepsilon\in(0,1/2]$

$$\|u_{h,r}-u_*\|\leqslant\|S_{h,r}P_h(u_0-u_*)\|+c(\varepsilon^{-1}+r^{1/2}\chi_{h,\varepsilon}^{1/2})\|(I-P_h)u_*\|+\gamma r\delta. \tag{30}$$

The Banach-Steinhaus theorem gives $\|S_{h,r}P_h(u_0-u_*)\|\to 0$ $(r\to\infty,\ h\to 0)$, hence the choice of r by (25) guarantees $u_{h,r}\to u_*$ $(\delta\to 0,\ h\to 0)$. Let (26) hold. Then $\|(I-P_h)u_*\|\leqslant\varrho\xi_{h,p}$ and (24) with $t=0$ gives

$$\|S_{h,r}P_h(u_0-u_*)\|\leqslant\varrho\|S_{h,r}P_hA^p\|\leqslant c\varrho[r^{-\min(p,p_0)}+\varrho\xi_{h,p}]. \tag{31}$$

From (30) and (31) it follows that the proposed choice of r gives estimates (27), (28) for $p\leqslant p_0$ and $p>p_0$, respectively. From (28) with $\varepsilon=1/2$ and the inequality

$$\chi_{h,\varepsilon}^p=\inf_{\alpha>0}(\xi_{h,\alpha}(1+\varepsilon^2))^{p/\alpha}\leqslant\xi_{h,p}(1+\varepsilon^2)\leqslant 5/4\xi_{h,p} \tag{32}$$

with $p=p_0$ we get (29). Theorem 3 is proved.

For methods 6′)—8′) Theorems 3, 5 (see the next Section) hold with p_1 instead of p_0.

Note that the convergence $u_{h,r}\to u_*$ $(\delta\to 0,\ h\to 0)$ was proved in [4, 1] in conditions similar to (25). In the process $\delta\to 0$, $h\to 0$ we assume that $r\chi_{h,\varepsilon}\leqslant c$, in [4] $r\xi_{h,1}\leqslant c$, and in [1] $r\inf_{\alpha>0}\xi_{h,\alpha}^{1/\alpha}2^{1/(4\alpha^2)}\leqslant c$ was assumed.

7. A-posteriori choice of r. Let $\varepsilon\in(0,1/2]$ and

$$w_{h,r}=T_{h,r}P_hu_0+g_{2,r}(A_h)P_hf_\delta,\qquad T_{h,r}=I-A_hg_{2,r}(A_h).$$

We use the following rules for the choice of r.

Rule I. Let $1<b_0\leqslant b$. If $\|A_hw_{h,0}-P_hf_\delta\|\leqslant b_0\delta$, take $r=0$. Otherwise choose $0<r\leqslant\bar r:=\chi_{h,\varepsilon}^{-1}$ such that

$$b_0\delta\leqslant\|A_hw_{h,r}-P_hf_\delta\|\leqslant b\delta. \tag{33}$$

If there is no $r\leqslant\bar r$ such that (33) holds, choose $r=\bar r$.

Rule II. Let $1<b$, $0<\Theta<1$. If $\|A_hw_{h,0}-P_hf_\delta\|\leqslant b\delta$, take $r=0$. Otherwise choose $0<r\leqslant\bar r$ such that there is a $r'\in[\Theta r, r]$ with

$$\|A_hw_{h,r}-P_hf_\delta\|\leqslant b\delta\leqslant\|A_hw_{h,r'}-P_hf_\delta\|. \tag{34}$$

If there is no $r\leqslant\bar r$ such that (34) holds, choose $r=\bar r$ or $r=[\bar r]$, where $[\bar r]$ is the largest integer, not greater than $\bar r$.

Lemma 4. *Let* $g_{2,r}(\lambda)$ *satisfy* (2), (22), $g_r(\lambda)$ *and* $g_{2,r}(\lambda)$ *satisfy*

A1) *the functions* $r\to|1-\lambda g_r(\lambda)|$, $r\to|1-\lambda g_{2,r}(\lambda)|$ *are decreasing* $(\forall\lambda\in[0,a])$,

A2) *for* $\lambda\in[0,a]$, $r>0$ *the function* $r\to g_r(\lambda)$ *is continuously differentiable and it holds that* $\partial g_r(\lambda)/\partial r\leqslant\gamma'(1-\lambda g_{2,r}(\lambda))$, $\gamma'=$const.

If $\|A_hw_{h,r_\delta}-P_hf_\delta\|\leqslant b\delta$, *then for every* $\alpha\in(0,1/2]$, $\varepsilon\in(0,1/2]$ *it holds that*

$$\|S_{h,r_\delta}P_h(u_0-u_*)\|\leqslant\inf_{r\geqslant 0}\psi_\alpha(r),$$

$$\psi_\alpha(r):=\|S_{h,r}P_h(u_0-u_*)\|+\bar\gamma c_2(\varepsilon^{-1}\alpha^{-1}r^\alpha\xi_{h,\alpha}+2r^{1/2}\chi_{h,\varepsilon}^{1/2})\|(I-P_h)u_*\|+$$
$$+\bar\gamma(b+1)r\delta,$$

where $\bar\gamma>\gamma'$ *and* c_2 *is given in* (23).

Proof. From the equality

$$A_hw_{h,r}-P_hf_\delta=T_{h,r}[A_h(u_0-u_*)-P_hA(I-P_h)u_*]+T_{h,r}P_h(f_\delta-f)$$

due to $\|T_{h,r}\|\leqslant\sup_{0\leqslant\lambda\leqslant a}|1-\lambda g_{2,0}(\lambda)|=1$ (see A1), (2)) we have

$$\|A_hw_{h,r}-P_hf_\delta\|-\delta\leqslant\|T_{h,r}[A_h(u_0-u_*)-P_hA(I-P_h)u_*]\|\leqslant$$
$$\leqslant\|A_hw_{h,r}-P_hf_\delta\|+\delta. \tag{35}$$

Hence $\|A_hw_{h,r_\delta}-P_hf_\delta\|\leqslant b\delta$ implies

$$\|T_{h,r_\delta}[A_h(u_0-u_*)-P_hA(I-P_h)u_*]\|\leqslant(b+1)\delta. \tag{36}$$

Let $\alpha\in(0,1/2]$ be fixed. Since in $\psi_\alpha(r)$ the first two terms are decreasing and the third is increasing, it has a unique minimum point r_0. We show that $r_\delta\geqslant r_0$. If $r_0=0$, it is obvious. If $r_0\neq 0$, then denoting $v:=P_h(u_0-u_*)$ we have

$$\psi'_{\alpha}(r_0)=0.5\|S_{h,r_0}v\|^{-1}\frac{\partial}{\partial r}(\|S_{h,r}v\|^2)\big|_{r=r_0}+$$

$$+\bar{\gamma}c_2(\varepsilon^{-1}r_0^{\alpha-1}\xi_{h,\alpha}+r_0^{-1/2}\chi_{h,\varepsilon}^{1/2})\|(I-P_h)u_*\|+\bar{\gamma}(b+1)\delta=0. \quad (37)$$

Let $Q(\lambda)$ be the spectral family of projectors of operator A_h. Using condition A2) we have

$$-\frac{\partial(1-\lambda g_r(\lambda))^2}{\partial r}=2\lambda(1-\lambda g_r(\lambda))\cdot\frac{\partial(g_r(\lambda))}{\partial r}\leqslant$$
$$\leqslant 2\gamma'\lambda(1-\lambda g_{2,r}(\lambda))(1-\lambda g_r(\lambda))$$

and so

$$-\frac{\partial}{\partial r}(\|S_{h,r}v\|^2)=-\int_0^{\|A_h\|}\frac{\partial}{\partial r}(1-\lambda g_r(\lambda))^2 d\langle Q(\lambda)v,v\rangle\leqslant$$

$$\leqslant 2\gamma'\int_0^{\|A_h\|}\lambda(1-\lambda g_{2,r}(\lambda))(1-\lambda g_r(\lambda))\cdot d\langle Q(\lambda)v,v\rangle=2\gamma'(A_hT_{h,r}v,S_{h,r}v)\leqslant$$
$$\leqslant 2\gamma'\|T_{h,r}A_hv\|\,\|S_{h,r}v\|.$$

From the last inequality, (23) and (37) we have

$$\|T_{h,r_0}[A_h(u_0-u_*)-P_hA(I-P_h)u_*]\|\geqslant\|T_{h,r_0}A_h(u_0-u_*)\|-$$

$$-\|T_{h,r_0}P_hA(I-P_h)\|\,\|(I-P_h)u_*\|\geqslant-\frac{1}{2\gamma'}\|S_{h,r_0}v\|^{-1}\frac{\partial}{\partial r}(\|S_{h,r}v\|^2)\big|_{r=r_0}-$$
$$-c_2(\varepsilon^{-1}r_0^{\alpha-1}\xi_{h,\alpha}+r_0^{-1/2}\chi_{h,\varepsilon}^{1/2})\|(I-P_h)u_*\|>(b+1)\delta.$$

The last inequality with (36) and condition A1) implies that $r_\delta>r_0$. Using once more condition A1) we get the assertion of Lemma 4:

$$\|S_{h,r_\delta}P_h(u_0-u_*)\|\leqslant\|S_{h,r_0}P_h(u_0-u_*)\|\leqslant\psi_\alpha(r_0)=\inf_{r\geqslant 0}\psi_\alpha(r).$$

Theorem 5. *Let $g_r(\lambda)$ satisfy* (2), (3), *$g_{2,r}(\lambda)$ satisfy* (2), (22) *with $p_2\in(1,p_0+1]$ and they both satisfy* A1), A2). *Let* $\varepsilon\in(0,1/2]$. *Let $r=r(\delta,h)$ be chosen by Rule* I *or Rule* II. *If $P_h\to I$ pointwise and* $\xi_{h,1}\to 0$ $(h\to 0)$, *then* $u_{h,r}\to u_*$ $(\delta\to 0,\ h\to 0)$. *If* (26) *holds, then*

$$r(\delta,h)\leqslant c((\delta/\varrho)^{-1/(p'+1)}+\varepsilon^{-1}(\varrho/\delta)\xi_{h,p}),\quad p'=\min(p,p_2-1), \quad (38)$$

$$\|u_{h,r}-u_*\|\leqslant c[(\varrho\delta^p)^{1/(p+1)}+\varepsilon^{-1}\varrho\xi_{h,p}],\qquad (p\leqslant p_2-1), \quad (39)$$

$$\|u_{h,r}-u_*\|\leqslant c[(\varrho\delta^{p_2-1})^{1/p_2}+\varrho(\varepsilon^{-1}\xi_{h,p}+\chi_{h,\varepsilon}^{\min(p,p_0)})+$$
$$+\varrho(\varepsilon^{-1}\alpha^{-1}\xi_{h,\alpha}\xi_{h,p})^{p_0/(p_0+\alpha)}],\quad(\forall\alpha\in(0,1/2])\quad(p>p_2-1). \quad (40)$$

Proof. For $r\leqslant\bar{r}$ we have from (30)

$$\|u_{h,r}-u_*\|\leqslant\|S_{h,r}P_h(u_0-u_*)\|+c\varepsilon^{-1}\|(I-P_h)u_*\|+\gamma r\delta. \quad (41)$$

We have $\|(I-P_h)u_*\|\to 0$ $(h\to 0)$; it is estimated by $\varrho\xi_{h,p}$, if (26) holds. Let Rule I or II give $r(\delta,h)<\bar{r}$. Then $\|A_hw_{h,r(\delta,h)}-P_hf_\delta\|\leqslant b\delta$ and Lemma 4 gives $\|S_{h,r(\delta,h)}P_h(u_0-u_*)\|\leqslant\inf_{r\geqslant 0}\psi_\alpha(r)$ $(\forall\,\alpha\in(0,1/2])$. The term $\psi_\alpha(r)$ may be considered in a similar way as the right-hand side of estimate (30). As in the proof of Theorem 3 we get .for $\|S_{h,r(\delta,h)}P_h(u_0-u_*)\|$ the convergence and under condition (26) the estimate

$$\|S_{h,r(\delta,h)}P_h(u_0-u_*)\|\leqslant\inf_{r\geqslant 0}[c\varrho r^{-\min(p,p_0)}+\gamma(b+1)r\delta+$$
$$+(c+\bar{\gamma}c_2(\varepsilon^{-1}\alpha^{-1}r^\alpha\xi_{h,\alpha}+2r^{1/2}\chi_{h,\varepsilon}^{1/2}))\varrho\xi_{h,p}]\quad(\forall\,\alpha\in(0,1/2],\quad\forall\,\varepsilon\in(0,1/2]). \quad (42)$$

If $p\leqslant p_0$, the choice of $\alpha=\min(p,1/2)$, $\varepsilon=1/2$ and r as in Theorem 3 gives the estimate in the right-hand side of (27). If $p>p_0$, the minimizing in (42) over r yields

$$\|S_{h,r(\delta,h)}P_h(u_0-u_*)\|\leqslant c[(\varrho\delta^{p_0})^{1/(p_0+1)}+\varrho(\chi_{h,\varepsilon}^{1/2}\xi_{h,p})^{2p_0/(2p_0+1)}+$$
$$+\varrho(\varepsilon^{-1}\alpha^{-1}\xi_{h,\alpha}\xi_{h,p})^{p_0/(p_0+\alpha)}],$$

which agrees with (40) due to $(\chi_{h,\varepsilon}^{1/2}\xi_{h,p})^{2p_0/(2p_0+1)}\leqslant\chi_{h,\varepsilon}^{p_0}+\xi_{h,p}$. If Rule I or II gives $r(\delta,h)=\bar{r}$, the proof of Theorem 3 yields the convergence $\|S_{h,r(\delta,h)}P_h(u_0-u_*)\|\to 0$ $(\delta\to 0,\ h\to 0)$ and in case (26) the estimate

$$\|S_{h,r(\delta,h)}P_h(u_0-u_*)\|\leqslant c\varrho[\chi_{h,\varepsilon}^{\min(p,p_0)}+\xi_{h,p}],$$

which, due to (32), agrees with (27), (40).

Consider the last term in (41). Let Rule I give $r=r(\delta,h)>0$. Then from (23) and the left-hand sides of (33), (35) we have, due to $r\leqslant\bar{r}$,

$$(b_0-1)r\delta\leqslant r\|T_{h,r}A_h(u_0-u_*)\|+r\|T_{h,r}P_hA(I-P_h)\|\,\|(I-P_h)u_*\|\leqslant$$
$$\leqslant r\|T_{h,r}A_h(u_0-u_*)\|+c\varepsilon^{-1}\|(I-P_h)u_*\|\to 0\qquad(\delta\to 0,\ h\to 0). \quad (43)$$

We used the convergence $r\|T_{h,r}A_h(u_0-u_*)\|\to 0$ $(r\to\infty)$, following from (22) and the Banach-Steinhaus theorem. Let (26) hold. Using (43) and (24) with $T_{h,r}$ instead of $S_{h,r}$ we get

$$(b_0-1)r\delta\leqslant r\|T_{h,r}A_hP_hA^p\|\varrho+c\varepsilon^{-1}\varrho\xi_{h,p}\leqslant c\varrho(r^{-p'}+\varepsilon^{-1}\xi_{h,p})$$

with $p'=\min(p,p_2-1)$. From here we have in case $r\geqslant(\delta/\varrho)^{-1/(p'+1)}$

$$r\delta\leqslant c[(\varrho\delta^{p'})^{1/(p'+1)}+\varepsilon^{-1}\varrho\xi_{h,p}]. \quad (44)$$

In case $r<(\delta/\varrho)^{-1(p'+1)}$ (44) is trivial. Inequality (44) implies that (38) holds and the last term in (41) has also an estimate, asserted in (39) and (40). Theorem 5 for Rule I is proved. For Rule II in a similar way we get analogues of (43), (44) with r' instead of r, hence the relation $r\leqslant r'/\Theta$ completes the proof.

Corollary. *The assertions of Theorem* 5 *are valid:*
1) *for methods* 1′), 2′) *with $p_2=\infty$, if $w_{h,r}=u_{h,r}$ in Rules* I, II *is used;*
2) *for method* 3′) *with $p_2=2$, if $w_{h,r}=u_{h,r,2}$ (see method* 4′)) *or $w_{h,r}=(u_{h,r}-\Theta u_{h,\Theta r})/(1-\Theta)$ in Rules* I, II *is used;*
3) *for method* 4′) *with $p_2=m$, if $w_{h,r}=u_{h,r}$ in Rules* I, II *is used, and with $p_2=m+1$, if $w_{h,r}=u_{h,r,m+1}$ or $w_{h,r}=(u_{h,r}-\Theta^m u_{h,\Theta r})/(1-\Theta^m)$ is used;*
4) *for method* 5′) *(case $q>1$) with $p_2=q$, if $w_{h,r}=u_{h,r}$ in Rules* I, II *is used;*
5) *for method* 6′) *(case $n=2$) with $p_2=m+1$, if $w_{h,r}=v_{h,r}$ in Rules* I, II *is used (the case $n>2$ needs an additional analysis);*
6) *for method* 7′) *with $p_2=qn$, if $w_{h,r}=v_{h,r}$ in Rules* I, II *is used;*
7) *for method* 8′) *with $p_2=qnm$, if $w_{h,r}=v_{h,r}$ in Rules* I, II *is used.*

Note that the choice of h by the rule $\xi_{h,\lambda}^{1/\lambda}\leqslant c\delta$ with $\lambda=(p_2-1)/p_2$ for $p_2<\infty$ and $\lambda=1$ for $p_2=\infty$ ensures that if (26) holds with $p\leqslant p_2-1$ (and with unknown p and ϱ), then under the assumptions of Theorem 5, it holds $\|u_{h,r}-u_*\|\leqslant c\delta^{p/(p+1)}$ (see [1]).

8. Applications to integral equations. Let $Au(t) \equiv \int_0^1 \mathcal{K}_1(t,s)u(s)\,ds$, where $\mathcal{K}_1(t,s) = \mathcal{K}_1(s,t)$ is such that $A = A^* \geqslant 0$ in $L_2 = L_2[0,1]$. The corresponding iterated kernel is $\mathcal{K}_n(t,s) = \mathcal{K}_n(s,t) = \int_0^1 \mathcal{K}_{n-1}(\sigma,t)\mathcal{K}_1(\sigma,s)\,d\sigma$, $n = 2, 3, \ldots$. Let l_n, $n = 1, 2, \ldots$ be the greatest numbers for which

$$\int_0^1 \int_0^1 \left| \frac{\partial^{l_n} \mathcal{K}_n(t,s)}{\partial t^{l_n}} \right|^2 dt\,ds < \infty, \qquad n = 1, 2, \ldots. \tag{45}$$

If (45) holds for every l_n, define $l_n = \infty$.

Let $\mathcal{R}(P_h)$ be a spline space with degree $k-1$. Then (see [1])

$$\xi_{h,p} = \| (I - P_h) A^p \| = O(h^{\min(pl_n/n,k)}) \qquad (p \leqslant n),$$

hence in case $l_1 < \infty$ with $\beta = \min(k/l_1, 1)$ we have

$$\chi_{h,\varepsilon} := \inf_{\alpha > 0} (\xi_{h,\alpha}(1+\varepsilon^2))^{1/\alpha} \leqslant [\xi_{h,\beta}(1+\varepsilon^2)]^{1/\beta}] = O(h^{l_1}).$$

If $l_1 = \infty$, then $\chi_{h,\varepsilon} = O[h^k(1+\varepsilon^2)]^{1/\alpha}$ for every $\alpha > 0$.

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PARAMEETRI VALIKUST REGULARISEERITUD RITZI-GALJORKINI MEETODIS

On vaadeldud lineaarse enesekaasse mittenegatiivse operaatoriga mittekorrektset ülesannet Hilberti ruumis. Võrrand on diskretiseeritud Ritzi-Galjorkini meetodiga ja seejärel regulariseeritud, kasutades kas iteratsioonimeetodeid või Lavrentjevi meetodit koos selle modifikatsioonidega. Viimasel juhul on näidatud, et lähendite asemel nende lineaarkombinatsiooni kasutamine võimaldab suurendada täpsust. Nii aprioorse kui ka aposterioorse regularisatsiooniparameetri valiku jaoks on antud koonduvustingimused ja veahinnangud, mis täpsustavad seniseid tulemusi.

Уно ХЯМАРИК

О ВЫБОРЕ ПАРАМЕТРА В РЕГУЛЯРИЗОВАННОМ МЕТОДЕ РИТЦА—ГАЛЕРКИНА

В гильбертовом пространстве H рассматривается линейное уравнение $Au = f$, $A = A^* \geqslant 0$. Вместо $f \in \mathcal{R}(A)$ задан $f_\delta \in H$, $\|f_\delta - f\| \geqslant \delta$. Уравнение дискретизируется методом Ритца—Галеркина $A_h u_h = P_h f_\delta$, $A_h = P_h A P_h$, $u_h \in \mathcal{R}(P_h)$, где $P_h (h > 0)$ — ортопроекторы в H. Регуляризованное решение $u_{h,r}$ для этого уравнения строится в виде (12), используя, например, методы итерации или метод Лаврентьева с модификациями. В последнем случае показывается, что $\sum_{i=1}^{n} d_i u_{h,r_i}$ с подходящими d_i, $r_i \in \mathbf{R}$, $i = 1, \ldots n$ сходится к гладкому решению быстрее, чем $u_{h,r}$. Рассматривается как априорный, так и апостериорный выбор параметра регуляризации r. Даются условия сходимости и оценки погрешности, уточняющие известные результаты. Результаты прилагаются к решению интегральных уравнений первого рода при помощи сплайнов.