Ill-Posed Problems in Natural Sciences, pp. 24 – 28
A. Tikhonov (Ed.)
1992 VSP/TVP

ON THE DISCRETIZATION ERROR IN REGULARIZED PROJECTION METHODS WITH PARAMETER CHOICE BY DISCREPANCY PRINCIPLE

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ABSTRACT

Regularized projection methods for solving linear ill-posed problems are considered. The regularization parameter is chosen by the (modified) discrepancy principle. Our error estimate has the optimal order with respect to the data errors and our estimate of discretization error is better than those given in earlier papers.

1. REGULARIZED PROJECTION METHODS

Consider the equation

$$Au = f, \quad f \in \mathcal{R}(A) \neq \overline{\mathcal{R}(A)}, \tag{1}$$

where $A \in \mathcal{L}(H, F)$, H, F are Hilbert spaces. Assume that only $f_\delta \in F$ is available with $\|f_\delta - f\| \leq \delta$. Let $h > 0$ be the discretization step and P_h, Q_h be the orthoprojectors in H, F, respectively, with $P_h \to I$, $Q_h \to I$ pointwise in H, F respectively, $\|A(I-P_h)\| \to 0$, $\|(I-Q_h)A\| \to 0$ $(h \to 0)$. The projection method for (1) has the form $A_h u_h = Q_h f_\delta$, $A_h = Q_h A P_h$, $u_h \in \mathcal{R}(P_h)$. We regularize this equation using a Borel measurable function g_r: $[0, a] \to \mathbf{R}$ $(r \geq 0)$ with the following properties for $r > 0$:

1) $|g_r(\lambda)| \leq \gamma r$ $(0 \leq \lambda \leq a,\ \gamma = \text{const})$,
2) $\lambda^p |1 - \lambda g_r(\lambda)| \leq \gamma_p r^{-p}$ $(0 \leq \lambda \leq a,\ 0 \leq p \leq p_0,\ p_0 > 0,\ \gamma_p = \text{const})$,
3) $r \to g_r(\lambda)$ is continuous and there holds $\frac{\partial(g_s(\lambda))}{\partial s}\Big|_{s=r} \leq \gamma' \beta_r(\lambda)(1 - \lambda g_r(\lambda))$, $\gamma' = \text{const}$, where $\beta_r(\lambda) = 1$ for $p_0 = \infty$, $\beta_r(\lambda) = (1 - \lambda g_r(\lambda))^{1/p_0}$ for $p_0 < \infty$,
4) $r \to |1 - \lambda g_r(\lambda)|$ is decreasing for any $\lambda \geq 0$.

Let $u_0 \in H$ and let u_* be the solution of (1), nearest to u_0. We find

$$u_{h,r} = (I - g_r(A_h^* A_h) A_h^* A_h) P_h u_0 + g_r(A_h^* A_h) A_h^* f_\delta, \tag{2}$$

assuming that $\|A\|^2 \leq a$. In the special case

$$F = H, \quad A = A^* \geq 0, \quad Q_h = P_h \tag{3}$$

we assume that $\|A\| \leq a$ and use the regularized Ritz–Galerkin method

$$u_{h,r} = (I - g_r(A_h)A_h)P_h u_0 + g_r(A_h)P_h f_\delta. \tag{4}$$

Examples of regularization methods of the form (2), (4) with the corresponding functions g_r, satisfying 1)–4) are the ordinary Lavrentiev and Tikhonov methods ($p_0 = 1$), their iterated versions of order m ($p_0 = m$), explicit and implicit iteration methods ($p_0 = \infty$) (see (Plato and Vainikko, 1989, 1990; Gfrerer, 1987)).

2. RULES FOR CHOOSING r

Let $B_{h,r} = B'_{h,r} = I$ if $p_0 = \infty$, $B_{h,r} = \left(I - A_h A_h^* g_r(A_h A_h^*)\right)^{1/(2p_0)}$ if $p_0 < \infty$. In the case (3) define $B'_{h,r} = \left(I - g_r(A_h)A_h\right)^{1/p_0}$ if $p_0 < \infty$.

RULE 1. *Let* $b_2 \geq b_1 \geq 1$, $0 < \Theta \leq 1$. *If* $\|B_{h,r}(A_h u_0 - Q_h f_\delta)\| \leq b_1\delta$, *choose* $r = 0$. *Otherwise choose* $0 < r \leq \bar{r} := \sup_{q>0}\{(2q)^{1/q}\|(I - P_h)|A|^q\|^{-2/q}\}$ ($|A| = (A^*A)^{1/2}$) *such that*

$$\left\|B_{h,r}(A_h u_{h,r} - Q_h f_\delta)\right\| \leq b_2\delta, \quad \exists s \in [\Theta r, r]\colon \left\|B_{h,s}(A_h u_{h,s} - Q_h f_\delta)\right\| \geq b_1\delta. \tag{5}$$

If no $r \leq \bar{r}$ *exists such that (5) holds, choose* $r = \bar{r}$ *or* $r = \operatorname{int}(\bar{r}) + 1$.

Rule 2 = Rule 1, where $B_{h,r}$ is replaced by I.
If (3) holds, then we choose in r approximation (4) according to Rules 1′, 2′ which we obtain from Rules 1, 2 by using the substitutions $B_{h,r} \to B'_{h,r}$, $\bar{r} \to (\bar{r})^{1/2}$, $Q_h \to P_h$.

3. CONVERGENCE AND RATE OF CONVERGENCE

THEOREM 1. **a)** *Let* r *in (2) be chosen by Rule 1. Then* $u_{h,r} \to u_*$ ($\delta \to 0$, $h \to 0$). *If*

$$u_* = |A|^p z, \quad \|z\| \leq \rho, \quad u_* - u_0 = |A|^p v, \quad \|v\| \leq \rho, \tag{6}$$

with $p \leq 2p_0$, *then*

$$\|u_{h,r} - u_*\| \leq c\left\{\rho^{1/(p+1)}\delta^{p/(p+1)} + \rho\|(I - P_h)|A|^p\| + \rho e(Q_h)\right\}, \tag{7}$$

where

$$e(Q_h) = \begin{cases} O\left(\|(I - Q_h)|A^*|^\mu\|^{\min\{p/\mu,2\}}\right) \ \forall \mu,\ 0 < \mu \leq 1,\ \mu \neq p/2 & \text{for } p \leq 2, \\ O\left(\left(1 + \log\|(I - Q_h)|A^*|^{p/2}\|\right)\|(I - Q_h)|A^*|^{p/2}\|^2\right) & \text{for } p \leq 2, \\ O\left(\|(I - Q_h)A\|\ \|(I - Q_h)|A^*|^{p-1}\|\right) & \text{for } p \geq 2. \end{cases}$$

b) *Let $p_0 > 1/2$ and r in (2) be chosen by Rule 2. Then $u_{h,r} \to u_*$ $(\delta \to 0,\ h \to 0)$. If (6) holds with $p \leq 2p_0 - 1$, then (7) holds.*

THEOREM 2. *Let (3) hold.* a) *Let r in (4) be chosen by Rule 1'. Then $u_{h,r} \to u_*$ $(\delta \to 0,\ h \to 0)$. If (6) holds with $p \leq p_0$, then (7) holds with the term $\rho e(Q_h)$ omitted.*

b) *Let* $p_0 \geq 1$ and *r in (4) be chosen by Rule 2'. Then $u_{h,r} \to u_*$ $(\delta \to 0,\ h \to 0)$. If (6) holds with $p \leq p_0 - 1$, then (7) holds with the term $\rho e(Q_h)$ omitted.*

The proof of Theorem 2 can be found in (Hämarik, 1991).

4. CHOICE OF h

A reasonable choice of h in (2) is determined by the rule

$$\left\|(I - P_h)|A|^\lambda\right\|^{1/\lambda} + \left\|(I - Q_h)|A^*|^\mu\right\|^{1/\mu} \approx \delta, \tag{8}$$

and in (4) by the rule $\|(I - P_h)A^\lambda\|^{1/\lambda} \approx \delta$ (provided that (3) holds). Table 1 presents the values of λ and μ depending on p_0 and the rule for choosing r. The choice of h guarantees that if (6) holds (with an unknown p) then $\|u_{h,r} - u_*\| \leq c\delta^{p/(p+1)}$ for $p \leq \bar{p}$, where the values of $\bar{p}$ are given in the last row of Table 1.

Table 1. Values of λ and μ in rules for choice of h

approx.	(4) (provided(3))			(2)		
p_0	∞	$p_0 < \infty$	$1 < p_0 < \infty$	∞	$p_0 < \infty$	$1/2 < p_0 < \infty$
r by Rule	2'	1'	2'	2	1	2
λ	1	$\frac{p_0}{p_0+1}$	$\frac{p_0-1}{p_0}$	1	$\frac{2p_0}{2p_0+1}$	$\frac{2p_0-1}{2p_0}$
μ				$\frac{1}{2}$	$\frac{p_0}{2p_0+1}$	$\frac{2p_0-1}{4p_0}$
$\bar{p}$	∞	p_0	$p_0 - 1$	∞	$2p_0$	$2p_0 - 1$

5. APPLICATION TO INTEGRAL EQUATIONS

Let $Au(t) = \int_0^1 \mathcal{K}(t,s)u(s)\,ds$,

$$\int_0^1\int_0^1 \left|\frac{\partial^{l_1}\mathcal{K}(t,s)}{\partial s^{l_1}}\right|^2 dt\,ds < \infty, \qquad \int_0^1\int_0^1 \left|\frac{\partial^{l_2}\mathcal{K}(t,s)}{\partial t^{l_2}}\right|^2 dt\,ds < \infty,$$

A: $\mathrm{L}_2[0,1] \to \mathrm{L}_2[0,1]$. Let $\mathcal{R}(P_{h_1})$, $\mathcal{R}(Q_{h_2})$ be spline spaces of degrees $k_1 - 1$, $k_2 - 1$ and discretization step sizes h_1, h_2, respectively. Then

$$\left\|(I - P_{h_1})|A|^p\right\| = O(h_1^{\min\{pl_1, k_1\}}), \tag{9}$$

$$\|(I - Q_{h_2})|A^*|^p\| = O(h_2^{\min\{pl_2,k_2\}}), \tag{10}$$

if $p \leq 1$. Hence, (8) means $h_1^{\min\{l_1,k_1/\lambda\}} + h_2^{\min\{l_2,k_2/\mu\}} \approx \delta$ and in § 2 the value $h_1^{-2l_1}$ may be used instead of $\bar{r}$.

6. COMPARISON WITH EARLIER RESULTS

In (Plato and Vainikko, 1989, 1990), for (2) in the case (6), the error estimate

$$\|u_{h,r} - u_*\| \leq c\Big\{\rho^{1/(p+1)}\delta^{p/(p+1)} + \rho\|A(I-P_h)\|^{\min\{p,1\}} + \rho\|(I-Q_h)A\|^{\min\{p,2\}}\Big\} \tag{11}$$

was obtained and, for (4), estimate (11) was stated without the last term under the assumption that (3) and (6) hold. Note that estimates (7) and (11) for the regularized least squares method (the case $Q_hAP_h = AP_h$) hold without the last term and for the regularized least error method (the case $Q_hAP_h = Q_hA$) without the second term. Corresponding estimate (11) for the first of these methods was earlier given in (Groetsch et al., 1982; Groetsch, 1984; Gfrerer, 1987) and for the second method in (King and Neubauer, 1988; Neubauer, 1988). Note also that estimate (7) is always not worse that (11), but there are examples when

$$\|(I-P_h)|A|^p\| = O(h^{\max\{p,1\}}), \qquad \|A(I-P_h)\|^{\min\{p,1\}} = O(h^{\min\{p,1\}}),$$
$$e(Q_h) = O(h^{\max\{p,2\}}), \qquad \|(I-Q_h)A\|^{\min\{p,2\}} = O(h^{\min\{p,2\}}).$$

In the case $p < 1$ such estimates follow from (9) and (10) in view of the example of § 5 with $k_i = 1,\ l_i > i/p,\ \ (i = 1, 2)$. It is worth noting that rule (8) with $\lambda = \mu = 1$ (independently from p_0) was proposed for choosing h in (Plato and Vainikko, 1990).

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